PERGAMON

International Journal of Solids and Structures 37 (2000) 3775–3806

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Comments on loss of strong ellipticity in elastoplasticity

László Szabó*

Department of Applied Mechanics, Technical University of Budapest, Müegyetem rkp. 5, H-1111 Budapest, Hungary

Received 12 February 1997; in revised form 13 January 1999

Abstract

The loss of strong ellipticity is analyzed for a rate independent infinitesimal elastoplastic model. This local stability condition corresponds to the loss of positive definiteness of the symmetric part of the acoustic tensor. First, in the case of multisurface plasticity some expressions for the plastic hardening moduli are obtained for various bifurcation criteria. Next, explicit expressions for the critical plastic hardening modulus and the critical orientation are obtained in the case of single-surface plasticity (Hill type comparison solid). The analysis is based on a geometric method. Linear, isotropic elasticity, and a general nonassociative flow rule are assumed. However, the principal axes of the second order tensors of the plastic potential and yield surfaces gradients are coaxial. It is shown that, similar to the loss of ellipticity, the direction of the critical orientation is identical to one of the principal directions, except in the particular case where the gradient of the plastic potential and yield surfaces each have a double eigenvalue. In particular, explicit expression for the plastic hardening modulus, using the same geometric method, is also presented for the Raniecki type comparison solid. As an illustrative example, the critical orientation for the loss of strong ellipticity and the classical shear band localization (loss of ellipticity) are compared for axially-symmetric compression and tension.

Keywords: Loss of strong ellipticity; Shear band; Elastoplasticity

1. Introduction

In recent years, a large number of theoretical and computational contributions to the analysis of strain localization have been made. The conditions for the classical shear band localization in rate independent materials are now well-understood. The basic principles of classical discontinuous bifurcation were first discussed by Hadamard (1903), Hill (1962), Mandel (1962), and later by Rudnicki and Rice (1975) and Rice (1976). Based on Rice's work, the general formulation of localization of

* Tel.: +36-1-463-1488; fax: +36-1-463-3417.
E-mail address: szabo@mm.bme.hu (L. Szabó).

PII: S0020-7683(99)00062-1

deformation into shear bands in the small deformation range can be considered well established, and it was applied to predict the orientation of shear bands for various types of material models (see, e.g. Bardet, 1990; Bigoni and Hueckel, 1990a, 1990b, 1991; Ichikawa et al., 1990; Ottosen and Runesson, 1991a; Benallal, 1992). Shear bands in the large deformation range have also been extensively studied in the literature (e.g. Molenkamp, 1985; Zbib and Aifantis, 1988; Zbib, 1989, 1991, 1993; Yatomi et al., 1989; Duszek and Perzyna, 1991; Bigoni and Zaccaria, 1993; Szabó, 1994; Steinmann et al., 1997).

A standard method for calculating the critical shear band orientation and critical plastic hardening modulus is based on the vanishing of the determinant of the acoustic tensor, which is derived from the tangential constitutive stiffness tensor. This condition yields a set of plastic hardening moduli from which the critical value is obtained in a maximization procedure. This problem was solved by several authors, under the assumption of coaxiality of the gradients of plastic potential and the yield surface. A number of these works use the Lagrange multiplier method (see, e.g. Rudnicki and Rice, 1975; Bigoni and Hueckel, 1990a, 1991; Runesson et al., 1991; Ottosen and Runesson, 1991a, 1991b), or employ the geometrical method proposed by Benallal (1992) (see, e.g. Benallal and Comi, 1993, 1996; Perrin and Leblond, 1993; Szabó, 1994).

The criterion for the classical discontinuous bifurcation which corresponds to the loss of ellipticity is that the acoustic tensor has a zero eigenvalue. This instability refers to the onset of shear band localization. It is well known that when the elasto–plastic constitutive model is based on a nonassociative flow rule the constitutive tangent tensor and the acoustic tensors are not symmetric. In this case the real-valued eigenspectrum of the nonsymmetric acoustic tensor is bounded by the minimum and maximum eigenvalues of the symmetrized acoustic tensor, and the loss of positive definiteness of the symmetric part of the acoustic tensor will occur before the loss of ellipticity. This phenomenon corresponds to the loss of strong ellipticity, and it was analyzed by Ottosen and Runesson (1991a), Bigoni and Zaccaria (1992a, 1992b), Neilsen and Schreyer (1993) and, more recently, by Rizzi et al. (1996), and Szabó (1997). The condition of strong ellipticity is satisfied prior to the classical discontinuous bifurcation criterion.

In the context of loss of ellipticity (onset of shear bands), several explicit expressions for the critical plastic hardening modulus and critical shear band orientation have been obtained for rate independent associative and nonassociative elastoplasticity (see, e.g. some recent surveys: Needleman and Tvergaard, 1992; Petryk, 1997). Much less work has been done, however, for the case of loss of strong ellipticity. The critical plastic hardening modulus was calculated numerically for the nonassociative Drucker–Prager model, by Neilsen and Schreyer (1993). Bigoni and Zaccaria (1992a) have shown that the critical plastic hardening modulus for the loss of strong ellipticity is identical for the Raniecki and Hill type comparison solids. In addition, they have presented an analytical solution for the critical plastic hardening modulus in the case of the Raniecki comparison solids. In their work, first, a maximization problem is solved for an associative flow rule associated to the Raniecki type comparison solid. Then, a minimization problem is defined for the plastic hardening modulus determined from the maximization process, respect to a free parameter, which corresponds to the Raniecki's comparison solids. Although, this minimization problem can be solved easily, explicit expressions for the critical plastic hardening modulus and critical orientation similar to that ones, which are available in many papers for the case of loss of ellipticity (e.g. Runesson et al., 1991; Ottosen and Runesson, 1991a), were not presented.

The main purpose of this paper is to obtain a closed form solution for the loss of strong ellipticity for a Hill type comparison solid. The constitutive model considered here is based on a rate independent elastoplasticity theory with a general nonassociative flow rule in the small strain range.

In the first part, the loss of ellipticity and the loss of strong ellipticity are analyzed for a general multisurface plasticity model, which is referred to as the corner flow rules with of interaction several yield and plastic potential surfaces (as in Asaro, 1983; Koiter, 1960; Mandel, 1965; Ottosen and Ristinmaa, 1996; Steinmann, 1996; among others). In this context, some explicit expressions for the plastic hardening moduli are presented. Moreover, an expression of plastic hardening moduli obtained from the general bifurcation condition (loss of uniqueness) is also derived.

Then, an explicit expression for the critical plastic hardening modulus and critical orientation is derived for single-surface plasticity (Hill type comparison solid). In this case, linear, isotropic elasticity is assumed, and {the principal axes of the second order tensors of the plastic potential and yield surfaces gradients are coaxial, which limits the analysis to isotropic hardening only. This analysis is based on the geometric method which was first proposed by Benallal and Lemaitre (1991) and Benallal (1992). It is pointed out, however that, the geometric method used in this paper is slightly different from that of Benallal (1992).

In addition, explicit expression for the critical plastic hardening modulus in the case of Raniecki type comparison solid, using the same geometric method, is considered in Appendix B

Finally, as an illustrative example, the critical orientation for the loss of strong ellipticity and the classical shear band localization (loss of ellipticity) are compared for axially-symmetric compression and tension.

Regarding notation, tensors are denoted by bold-face characters, the order of which is indicated in the text. The tensor product is denoted by $\otimes$, and the following symbolic operations apply: $\mathbf{g}\cdot\mathbf{n}=g_i n_i$, $(\mathbf{A}\cdot\mathbf{n})_i=\mathbf{A}_{ij}n_j$, $(\mathbf{A}\cdot\mathbf{B})_{ij}=A_{ik}B_{kj}$, $\mathbf{A}{:}\mathbf{B}=A_{ij}B_{ij}$ and $(\mathbf{C}{:}\mathbf{A})_{ij}=C_{ijkl}A_{kl}$, with the summation convention over repeated indices. The superposed dot denotes the material time derivative, or rate. The superscripts T and -1 denote transpose and inverse, and the prefix tr indicates the trace. The symbol $\|\cdot\|$ is used to denote Euclidean norm. The fourth and second-order identity tensors are denoted by $\mathbf{I}$ and $\boldsymbol{\delta}$, respectively.

2. Constitutive relations and localization conditions

2.1. Constitutive relation

In the general case of rate independent multisurface elasto–plasticity, the widely used form of the constitutive relations in the small deformation range is expressed in terms of a relation between the stress rate and strain rate:

$$\dot{\boldsymbol{\sigma}} = \mathbf{D}^{\text{ep}}{:}\dot{\boldsymbol{\varepsilon}}, \tag{1}$$

where

$$\mathbf{D}^{\text{ep}} = \mathbf{D}^{\text{e}} - M^{\alpha\beta}\mathbf{D}^{\text{e}}{:}\mathbf{P}_{\alpha} \otimes \mathbf{Q}_{\beta}{:}\mathbf{D}^{\text{e}}; \quad \alpha,\beta \in [1,n] \tag{2}$$

is the incremental elasto–plastic stiffness tensor, and the summation convention with respect to repeated greek indices is adopted. Here, $\mathbf{D}^{\text{e}}$ is the fourth-order elasticity tensor,, $\mathbf{P}_{\alpha}$ and $\mathbf{Q}_{\beta}$ are the unit outward normals to the plastic potential and yield surfaces, and the matrix $M^{\alpha\beta}$ is defined by

$$(M^{\alpha\beta})^{-1} = m_{\alpha\beta} = h_{\alpha\beta} + \mathbf{Q}_{\alpha}{:}\mathbf{D}^{\text{e}}{:}\mathbf{P}_{\beta}, \tag{3}$$

where $h_{\alpha\beta}$ is the matrix of plastic moduli. In the present paper, the multisurfaces plasticity is referred to as the corner flow rules with of interaction several plastic potential and yield surfaces. Note that the inequalities for the stress rates specifying in the corner domain are not discussed in this paper.

There are a number of constitutive models in which the tangent operators are closely related to $\mathbf{D}^{\text{ep}}$ defined in
Eq. (2), for example, in the single crystal plasticity: Asaro (1983), Peirce (1983), Dao and Asaro (1996);

in the corner theories: Koiter (1960), Mandel (1965), Sewell (1973, 1974), Simo et al. (1988) and Reddy and Gültop (1995) (for associated plasticity), Ottosen and Ristinmaa (1996) (for non-associated plasticity); in general multisurface elastoplasticity: Rizzi et al. (1996), Sawischlewski et al. (1996) and Steinmann (1996).

In the single-surface plasticity ($\alpha = \beta = 1$), the tensor $\mathbf{D}^{ep}$ takes on the following simple form:

$$\mathbf{D}^{ep} = \mathbf{D}^{e} - \frac{\mathbf{D}^{e}:\mathbf{P} \otimes \mathbf{Q}:\mathbf{D}^{e}}{H + \mathbf{Q}:\mathbf{D}^{e}:\mathbf{P}}, \tag{4}$$

where H is the generalized plastic hardening modulus. Note that the tangent operator in this form defined above was considered, for example, by Prévost (1984), Bardet (1990) and Loret (1992).

2.2. The loss of ellipticity and loss of strong ellipticity conditions

2.2.1. Loss of ellipticity

Based on the analysis of acceleration waves (see, e.g. Hill, 1962; Mandel, 1962; Loret et al., 1990; Ottosen and Runesson, 1991b; Loret, 1992), or the general method of the shear band localization (see, e.g. Rudnicki and Rice, 1975; Rice, 1976; Rice and Rudnicki, 1980; Borré and Maier, 1989; Bigoni and Hueckel, 1991; Ottosen and Runesson, 1991a; Runesson et al., 1991; Neilsen and Schreyer, 1993), the second-order acoustic tensor $\mathbf{B}^{ep}$ for the constitutive relations Eq. (1) is given by

$$\mathbf{B}^{ep} = \mathbf{n} \cdot \mathbf{D}^{e} \cdot \mathbf{n} - M^{\alpha\beta}\mathbf{n} \cdot (\mathbf{D}^{e}:\mathbf{P}_{\alpha}) \otimes (\mathbf{Q}_{\beta}:\mathbf{D}^{e}) \cdot \mathbf{n}, \tag{5}$$

where $\mathbf{n}$ is unit vector normal to the front of the acceleration waves, or normal to the plane of discontinuity. The first term on the right-hand side of Eq. (5) can be defined as the elastic acoustic tensor

$$\mathbf{B}^{e} = \mathbf{n} \cdot \mathbf{D}^{e} \cdot \mathbf{n}, \tag{6}$$

and the quantities $\mathbf{n}\cdot(\mathrm{D}^{e}:\mathbf{P}_{\alpha})$ and $(\mathbf{Q}_{\beta}:\mathbf{D}^{e})\cdot\mathbf{n}$ by introducing two families of vectors, can be written as

$$\mathbf{a}_{\alpha} = \mathbf{n} \cdot (\mathbf{D}^{e}:\mathbf{P}_{\alpha})$$

and

$$\mathbf{b}_{\beta} = (\mathbf{Q}_{\beta}:\mathbf{D}^{e}) \cdot \mathbf{n}. \tag{7}$$

The loss of ellipticity criterion corresponds to the singularity of the acoustic tensor. When the acoustic tensor has a zero eigenvalue the determinant of $\mathbf{B}^{ep}$ equals zero

$$\det \mathbf{B}^{ep} = 0, \tag{8}$$

which is the necessary condition for the localization.

2.2.2. Loss of strong ellipticity

The loss of strong ellipticity corresponds to the loss of positive definiteness of the symmetric part of the acoustic tensor (Bigoni and Hueckel, 1991; Ottosen and Runesson, 1991a; Bigoni and Zaccaria, 1992a, 1992b; Neilsen and Schreyer, 1993; Rizzi et al., 1996). Equivalently, the loss of strong ellipticity is first satisfied when the determinant of the symmetrized acoustic tensor is equal to zero

$$\det \mathbf{B}^{ep}_{sym} = 0, \tag{9}$$

where $\mathbf{B}^{\mathrm{ep}}_{\mathrm{sym}}$ is defined by

$$\mathbf{B}^{\mathrm{ep}}_{\mathrm{sym}} = \mathbf{B}^{\mathrm{e}}_{\mathrm{sym}} - \frac{1}{2} M^{\alpha\beta} (\mathbf{a}_\alpha \otimes \mathbf{b}_\beta + \mathbf{b}_\alpha \otimes \mathbf{a}_\beta). \tag{10}$$

Note that, since the elasticity tensor $\mathbf{D}^{\mathrm{e}}$ is symmetric, the elastic acoustic tensor is identical its symmetric part, $\mathbf{B}^{\mathrm{e}} \equiv \mathbf{B}^{\mathrm{e}}_{\mathrm{sym}}$.

3. The plastic hardening moduli

3.1. Single-surface plasticity ($\alpha = \beta = 1$)

For single-surface plasticity an explicit expression for the plastic hardening modulus, using the conditions (Eq. (8) or Eq. (9)) can be easily derived. In this context several methods have been used (see e.g. Rice, 1976; Peirce, 1983; Molenkamp, 1985; Bigoni and Hueckel, 1991; Benallal, 1992; Bigoni and Zaccaria, 1992a; Doghri and Billardon, 1995; Steinmann, 1996). One of these (Steinmann, 1996) is based on the Sherman–Morrison formula, which will be applied in the present paper.

3.1.1. Loss of ellipticity

In this case, the acoustic tensor is defined by

$$\mathbf{B}^{\mathrm{ep}} = \mathbf{B}^{\mathrm{e}} - M \mathbf{a} \otimes \mathbf{b}, \tag{11}$$

where $M = 1/(H + \mathbf{Q}:\mathbf{D}^{\mathrm{e}}:\mathbf{P})$,

$$\mathbf{a} = \mathbf{n} \cdot (\mathbf{D}^{\mathrm{e}}:\mathbf{P})$$

and

$$\mathbf{b} = (\mathbf{Q}:\mathbf{D}^{\mathrm{e}}) \cdot \mathbf{n}. \tag{12}$$

The inverse of $\mathbf{B}^{\mathrm{ep}}$ can be done in an elementary way, similarly to the inversion of the elastoplastic constitutive tensor. The result takes the form

$$\mathbf{B}^{\mathrm{ep}-1} = \mathbf{B}^{\mathrm{e}-1} + \frac{\mathbf{B}^{\mathrm{e}-1} \cdot \mathbf{a} \otimes \mathbf{b} \cdot \mathbf{B}^{\mathrm{e}-1}}{\dfrac{1}{M} - \mathbf{b} \cdot \mathbf{B}^{\mathrm{e}-1} \cdot \mathbf{a}}. \tag{13}$$

When the condition $\det \mathbf{B}^{\mathrm{ep}} = 0$ holds, the tensor $\mathbf{B}^{\mathrm{ep}}$ cannot be inverted. Then, it follows from Eq. (13) that $1/M - \mathbf{b} \cdot \mathbf{B}^{\mathrm{e}-1} \cdot \mathbf{a} = 0$. From this condition, the plastic hardening modulus is given by

$$H^{\mathrm{le}} = (\mathbf{Q}:\mathbf{D}^{\mathrm{e}} \cdot \mathbf{n}) \cdot \mathbf{B}^{\mathrm{e}-1} \cdot (\mathbf{n} \cdot \mathbf{D}^{\mathrm{e}}:\mathbf{P}) - \mathbf{Q}:\mathbf{D}^{\mathrm{e}}:\mathbf{P}. \tag{14}$$

which has been derived by Rice (1976).

3.1.2. Loss of strong ellipticity

In the case of loss of strong ellipticity, the symmetrized acoustic tensor is expressed as

$$\mathbf{B}^{\mathrm{ep}}_{\mathrm{sym}} = \mathbf{B}^{\mathrm{e}}_{\mathrm{sym}} - \frac{1}{2} M (\mathbf{a} \otimes \mathbf{b} + \mathbf{b} \otimes \mathbf{a}) = \mathbf{A} - \frac{1}{2} M \mathbf{a} \otimes \mathbf{b}, \tag{15}$$

where

$$\mathbf{A} = \mathbf{B}^{\mathrm{e}}_{\mathrm{sym}} - \frac{1}{2} M \mathbf{b} \otimes \mathbf{a}. \tag{16}$$

In analogy with Eq. (13), we may define the inverse of $\mathbf{B}^{\mathrm{ep}}_{\mathrm{sym}}$ according to

$$\mathbf{B}^{\mathrm{ep}-1}_{\mathrm{sym}} = \mathbf{A}^{-1} + \frac{\mathbf{A}^{-1} \cdot \mathbf{a} \otimes \mathbf{b} \cdot \mathbf{A}^{-1}}{\dfrac{2}{M} - \mathbf{b} \cdot \mathbf{A}^{-1} \cdot \mathbf{a}} \tag{17}$$

where the inverse of $\mathbf{A}$ is defined by

$$\mathbf{A}^{-1} = \mathbf{B}^{\mathrm{e}-1}_{\mathrm{sym}} + \frac{\mathbf{B}^{-1}_{\mathrm{sym}} \cdot \mathbf{b} \otimes \mathbf{a} \cdot \mathbf{B}^{-1}_{\mathrm{sym}}}{\dfrac{2}{M} - \mathbf{a} \cdot \mathbf{B}^{-1}_{\mathrm{sym}} \cdot \mathbf{b}} \tag{18}$$

For the loss of strong ellipticity, the following implication holds:

$$\det \mathbf{B}^{\mathrm{ep}}_{\mathrm{sym}} = 0 \Longrightarrow \frac{2}{M} - \mathbf{b} \cdot \mathbf{A}^{-1} \cdot \mathbf{a} = 0. \tag{19}$$

Use of $\mathbf{A}^{-1}$ in this relation implies

$$\left(\frac{2}{M} - \mathbf{b} \cdot \mathbf{B}^{\mathrm{e}-1}_{\mathrm{sym}} \cdot \mathbf{a}\right)^2 - (\mathbf{a} \cdot \mathbf{B}^{\mathrm{e}-1}_{\mathrm{sym}} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{B}^{\mathrm{e}-1}_{\mathrm{sym}} \cdot \mathbf{b}) = 0. \tag{20}$$

From this condition, the plastic hardening modulus can be derived as

$$H^{\mathrm{lse}} = \frac{1}{2} \Big\{ (\mathbf{Q}{:}\mathbf{D}^{\mathrm{e}} \cdot \mathbf{n}) \cdot \mathbf{B}^{\mathrm{e}-1}_{\mathrm{sym}} \cdot (\mathbf{n} \cdot \mathbf{D}^{\mathrm{e}}{:}\mathbf{P}) + \sqrt{[(\mathbf{P}{:}\mathbf{D}^{\mathrm{e}} \cdot \mathbf{n}) \cdot \mathbf{B}^{\mathrm{e}-1}_{\mathrm{sym}} \cdot (\mathbf{n} \cdot \mathbf{D}^{\mathrm{e}}{:}\mathbf{P})][(\mathbf{Q}{:}\mathbf{D}^{\mathrm{e}} \cdot \mathbf{n}) \cdot \mathbf{B}^{\mathrm{e}-1}_{\mathrm{sym}} \cdot (\mathbf{n} \cdot \mathbf{D}^{\mathrm{e}}{:}\mathbf{Q})]} \Big\} - \mathbf{Q}{:}\mathbf{D}^{\mathrm{e}}{:}\mathbf{P}. \tag{21}$$

It follows from Eq. (20) that the discriminant in Eq. (21) is always greater than (or equal to) zero.

Note that an identical expression was derived by Bigoni and Zaccaria (1992a), albeit in a different way.

For loss of ellipticity and for loss of strong ellipticity the critical plastic hardening modulus corresponds to the solution of the following constrained maximization problems:

$$H^{\mathrm{le}}_{\mathrm{crit}} = \max_{n} (H^{\mathrm{le}}) \quad \text{and} \quad H^{\mathrm{lse}}_{\mathrm{crit}} = \max_{n} (H^{\mathrm{lse}}),$$

subject to $|\mathbf{n}| = 1$, respectively.

3.2. Multisurface plasticity ($\alpha, \beta \in [1, \mathrm{n}]$)

In the case of multisurface plasticity, the second-order acoustic tensor $\mathbf{B}^{\mathrm{ep}}$ (in Eq. (5)) has the form of a multiple rank one modification of the elastic acoustic tensor. The task is to find an explicit expression for the plastic hardening moduli by means of the determinant of $\mathbf{B}^{\mathrm{ep}}$ (for loss of ellipticity), and its symmetric part (for loss of strong ellipticity) is equal to zero.

A closed form expression to compute the determinant of a matrix with multiple rank one updates was presented by Steinmann (1996). In his study, a recursive application of the Sherman–Morrison formulas has been employed, and for the loss of ellipticity an explicit expression for the plastic hardening moduli in the cases of double, triple, and multisurface plasticity have been presented. Unfortunately, the application of the method presented by Steinmann (1996) for the case of loss of strong ellipticity is rather difficult because the number of rank one updates in the symmetrized acoustic tensor are twice as much.

However, using the following Lemma, we can obtain expressions for the plastic hardening moduli for various bifurcation criteria. Here we generalize the Sherman–Morrison formula to general case with a multiple rank one updates. Note that in several studies the Sherman–Morrison formula for the case of the sum of a regular matrix and one rank one updates, was employed, e.g., Szabó (1985), Doghri and Billardon (1995), Steinmann (1996), Rizzi et al. (1996) and Steinmann et al. (1997).

Lemma 1. Let $\mathbf{B}$ and $\mathbf{C}$ be two second-order tensors, and let $\mathbf{a}_\alpha$ and $\mathbf{b}_\beta$ be two families of vectors, where $\alpha, \beta \in [1, n]$. Define $\mathbf{B}$ as a sum of rank one updates of the non-singular tensor $\mathbf{C}$

$$\mathbf{B} = \mathbf{C} - M^{\alpha\beta}\mathbf{a}_\alpha \otimes \mathbf{b}_\beta, \tag{22}$$

where the matrix $M^{\alpha\beta}$ is invertible, and its inverse denoted by

$$(M^{\alpha\beta})^{-1} = m_{\alpha\beta} \Longrightarrow M^{\alpha\gamma}m_{\gamma\beta} = \delta^\alpha_\beta. \tag{23}$$

Then, the inverse of $\mathbf{B}$, is given by

$$\mathbf{B}^{-1} = \mathbf{C}^{-1} + G^{\alpha\beta}\mathbf{C}^{-1} \cdot \mathbf{a}_\alpha \otimes \mathbf{b}_\beta \cdot \mathbf{C}^{-1}, \tag{24}$$

where

$$G^{-1}_{\alpha\beta} = m_{\alpha\beta} - \mathbf{b}_\alpha \cdot \mathbf{C}^{-1} \cdot \mathbf{a}_\beta. \tag{25}$$

The proof of this lemma requires a simple verification, namely, multiplying $\mathbf{B}$ by $\mathbf{B}^{-1}$ defined by Eq. (25), we directly obtain $\boldsymbol{\delta}$.

Remark 1. In the simple case where the tensor $\mathbf{B}$ is defined in the form of single rank one update of $\mathbf{C}$ ($\mathbf{B}=\mathbf{C}-M\ \mathbf{a}\otimes\mathbf{b}$), the determinant of $\mathbf{B}$ can be expressed as (Szabó, 1994; Steinmann, 1996):

$$\det \mathbf{B} = \det \mathbf{C}(1 - M\mathbf{b} \cdot \mathbf{C}^{-1} \cdot \mathbf{a}).$$

The extension of this result to multiple rank one updates can be obtained analogously. The determinant of $\mathbf{B}$ (in Eq. (22)) is defined by

$$\det \mathbf{B} = \det \mathbf{C} \det \left[\delta^\alpha_\beta - M^{\alpha\delta}\left(\mathbf{b}_\delta \cdot \mathbf{C}^{-1} \cdot \mathbf{a}_\beta\right)\right], \tag{26}$$

which coincides with the result obtained by Steinmann (1996). When the matrix $M^{\alpha\delta}$ is nonsingular (and its inverse denoted by $m_{\alpha\delta}$), the expression defined above can be rewritten as

$$\det \mathbf{B} = \det \mathbf{C} \det(M^{\gamma\delta})\det(m_{\alpha\beta} - \mathbf{b}_\alpha \cdot \mathbf{C}^{-1} \cdot \mathbf{a}_\beta). \tag{27}$$

Since $\det \mathbf{C} \neq \mathbf{0}$ and $\det(M^{\gamma\delta}) \neq \mathbf{0}$, it is concluded that

$$\det \mathbf{B} = 0 \Longrightarrow \det(m_{\alpha\beta} - \mathbf{b}_\alpha \cdot \mathbf{C}^{-1} \cdot \mathbf{a}_\beta) = 0. \tag{28}$$

Let us examine the loss of ellipticity and loss of strong ellipticity using Lemma 1.

3.2.1. Loss of ellipticity

For the loss of ellipticity, from Eq. (24) (or Eq. (28)), using Eqs. (3) and (5)–(7), the following condition is immediately obtained:

$$\det \mathbf{B}^{\mathrm{ep}} = 0 \Longrightarrow \det(h_{\alpha\beta} + \mathbf{Q}_\alpha{:}\mathbf{D}^{\mathrm{e}}{:}\mathbf{P}_\beta - \mathbf{b}_\alpha \cdot \mathbf{B}^{\mathrm{e}-1} \cdot \mathbf{a}_\beta) = 0. \tag{29}$$

As mentioned above, a similar expression has been derived by Steinmann (1996). Note that Lemma 1 presented here allows one to obtain a result in simpler way. We also note that the same result was presented by Peirce (1983) in the context of single crystal plasticity.

3.2.2. Loss of strong ellipticity

Analogous to the case of single-surface plasticity, from the condition

$$\det \mathbf{B}^{\mathrm{ep}}_{\mathrm{sym}} = 0, \tag{30}$$

where $\mathbf{B}^{\mathrm{ep}}_{\mathrm{sym}}$ defined by Eq. (10), using Eq. (24), the following condition can be obtained

$$\det\left[h_{\alpha\beta} + \mathbf{Q}_\alpha{:}\mathbf{D}^{\mathrm{e}}{:}\mathbf{P}_\beta - \frac{1}{2}\mathbf{b}_\alpha \cdot \mathbf{B}^{\mathrm{e}-1}_{\mathrm{sym}} \cdot \mathbf{a}_\beta - \frac{1}{4}(\mathbf{b}_\alpha \cdot \mathbf{B}^{\mathrm{e}-1}_{\mathrm{sym}} \cdot \mathbf{b}_\gamma)N^{\gamma\delta}(\mathbf{a}_\delta \cdot \mathbf{B}^{\mathrm{e}-1}_{\mathrm{sym}} \cdot \mathbf{a}_\beta)\right] = 0. \tag{31}$$

The inverse of $N^{\gamma\delta}$ is defined by

$$N^{-1}_{\gamma\delta} = h_{\gamma\delta} + \mathbf{Q}_\gamma{:}\mathbf{D}^{\mathrm{e}}{:}\mathbf{P}_\delta - \frac{1}{2}\mathbf{b}_\gamma \cdot \mathbf{B}^{\mathrm{e}-1}_{\mathrm{sym}} \cdot \mathbf{a}_\delta, \tag{32}$$

here the indices α, β, γ and $\delta \in [1, n]$.

For convenience, we introduce the notations:

$$R_{\alpha\beta} = \frac{1}{2}\mathbf{b}_\alpha \cdot \mathbf{B}^{\mathrm{e}-1}_{\mathrm{sym}} \cdot \mathbf{a}_\beta - \mathbf{Q}_\alpha{:}\mathbf{D}^{\mathrm{e}}{:}\mathbf{P}_\beta,$$

$$B_{\alpha\beta} = \frac{1}{2}\mathbf{b}_\alpha \cdot \mathbf{B}^{\mathrm{e}-1}_{\mathrm{sym}} \cdot \mathbf{b}_\beta$$

and

$$A_{\alpha\beta} = \frac{1}{2}\mathbf{a}_\alpha \cdot \mathbf{B}^{\mathrm{e}-1}_{\mathrm{sym}} \cdot \mathbf{a}_\beta, \tag{33}$$

where $R_{\alpha\beta}$, $B_{\alpha\beta}$ and $A_{\alpha\beta}$ are the elements of the $n \times n$ matrices $\mathbf{R}$, $\mathbf{B}$ and $\mathbf{A}$, respectively. With these matrices, Eq. (31) takes the form

$$\det[\mathbf{h} - \mathbf{R} - \mathbf{B}(\mathbf{h} - \mathbf{R})^{-1}\mathbf{A}] = 0, \tag{34}$$

where $\mathbf{h} = [h_{\alpha\beta}]$ is the matrix of plastic hardening moduli.

Remark 2. It is important to note that Lemma 1 given above is also valid when $\mathbf{B}$ and $\mathbf{C}$ are fourth-order tensors, and when $\mathbf{a}_\alpha$ and $\mathbf{b}_\beta$ are two families of second-order tensors. As an illustration, we will give example of how to compute the critical plastic hardening moduli for the general bifurcation criterion. Some recent discussions of the bifurcations of elasto–plastic solids have shown that the loss of material stability corresponds to the loss of positive definiteness of the symmetric part of the elasto–plastic stiffness tensor (see, e.g. Ottosen and Runesson, 1991a; Neilsen and Schreyer, 1993; Ottosen and Ristinmaa, 1996). Equivalently, the necessary condition for a general bifurcation (loss of uniqueness) is first satisfied when the fundamental eigenvalue of the symmetric part of the fourth-order constitutive tangent tensor is equal to zero.

The symmetric part of the fourth-order elasto–plastic stiffness tensor, $\mathbf{D}^{\mathrm{ep}}$ (in Eq. (2)), can be expressed as

$$\mathbf{D}^{\mathrm{ep}}_{\mathrm{sym}} = \mathbf{D}^{\mathrm{e}} - \frac{1}{2} M^{\alpha\beta} \mathbf{D}^{\mathrm{e}}:(\mathbf{P}_\alpha \otimes \mathbf{Q}_\beta + \mathbf{Q}_\alpha \otimes \mathbf{P}_\beta):\mathbf{D}^{\mathrm{e}}; \quad \alpha,\beta \in [1,n]. \tag{35}$$

The general bifurcation criterion is first satisfied when the determinant of $\mathbf{D}^{\mathrm{ep}}_{\mathrm{sym}}$ is equal to zero (Neilsen and Schreyer, 1993):

$$\det \mathbf{D}^{\mathrm{ep}}_{\mathrm{sym}} = 0. \tag{36}$$

From this condition, using Eq. (31) with the following identifications, $\mathbf{a}_\alpha \rightarrow \mathbf{D}^{\mathrm{e}}:\mathbf{P}_\alpha$, $\mathbf{b}_\beta \rightarrow \mathbf{Q}_\beta:\mathbf{D}^{\mathrm{e}}$ and $\mathbf{B}^{\mathrm{e}-1}_{\mathrm{sym}} \rightarrow \mathbf{D}^{\mathrm{e}-1}$, the criterion for loss of uniqueness is defined by

$$\det\left[h_{\alpha\beta} + \frac{1}{2}\mathbf{Q}_\alpha:\mathbf{D}^{\mathrm{e}}:\mathbf{P}_\beta - \frac{1}{4}(\mathbf{Q}_\alpha:\mathbf{D}^{\mathrm{e}}:\mathbf{Q}_\gamma) Z^{\gamma\delta} (\mathbf{P}_\delta:\mathbf{D}^{\mathrm{e}}:\mathbf{P}_\beta)\right] = 0 \tag{37}$$

where α, β, γ and $\delta \in [1, n]$, and the inverse of $Z^{\gamma\delta}$ is defined by

$$Z^{-1}_{\gamma\delta} = h_{\gamma\delta} + \frac{1}{2}\mathbf{Q}_\gamma:\mathbf{D}^{\mathrm{e}}:\mathbf{P}_\delta. \tag{38}$$

It should be noted that a similar result is derived by Ottosen and Ristinmaa (1996) in a different way.

3.3. Example: Double-surface plasticity (α, $\beta \in [1, 2]$)

As an illustrative example of the method developed above, we will derive an explicit expression for the plastic hardening moduli in the case of double-surface plasticity. In this example the one parameter family of hardening moduli (see, e.g. Hutchinson, 1970; Asaro, 1983) is considered, which in the present case can be expressed as

$$h_{\alpha\beta} = HK_{\alpha\beta} \text{ or } \mathbf{h} = H\mathbf{K}, \tag{39}$$

where the matrix $\mathbf{K}$ does not depend on H explicitly.

3.3.1. Loss of ellipticity

For convenience, we introduce the $\mathbf{L}$ matrix of order 2 with coefficients

$$L_{\alpha\beta} = \mathbf{b}_\alpha \cdot \mathbf{B}^{\mathrm{e}-1} \cdot \mathbf{a}_\beta - \mathbf{Q}_\alpha:\mathbf{D}^{\mathrm{e}}:\mathbf{P}_\beta. \tag{40}$$

With this notation, the localization condition (29) now becomes:

$$\det(H\mathbf{K} - \mathbf{L}) = 0 \tag{41}$$

from which, using the Cayley–Hamilton theorem, we formulate a quadratic equation for the plastic hardening modulus:

$$(\det \mathbf{K})H^2 + [\mathrm{tr}(\mathbf{KL}) - \mathrm{tr}\,\mathbf{K}\,\mathrm{tr}\,\mathbf{L}]H + \det \mathbf{L} = 0. \tag{42}$$

3.3.2. Loss of strong ellipticity

In the case of double-surface plasticity, the matrices **R**, **B** and **A** defined by Eq. (33) are of order 2. The condition defined by Eq. (34) can be expressed as a fourth degree polynomial equation with respect to H:

$$a_4H^4 + a_3H^3 + a_2H^2 + a_1H + a_0 = 0, \tag{43}$$

where the parameters a_i ($i \in [0, 4]$) are defined by:

$$a_4 = (\det \mathbf{K})^2,$$

$$a_3 = 2 \det \mathbf{K}\{\mathrm{tr}(\mathbf{KR}) - \mathrm{tr}\,\mathbf{K}\,\mathrm{tr}\,\mathbf{R}\},$$

$$a_2 = 2 \det \mathbf{K} \det \mathbf{R} - (\mathrm{tr}\,\mathbf{R})^2\,\mathrm{tr}(\mathbf{BA}) - \mathrm{tr}(\mathbf{KBKA}) + \mathrm{tr}\,\mathbf{K}\{\mathrm{tr}(\mathbf{BKA}) + \mathrm{tr}(\mathbf{KBA})\} + \{\mathrm{tr}(\mathbf{K},\mathbf{R}) - \mathrm{tr}\,\mathbf{K}\,\mathrm{tr}\,\mathbf{R}\}^2,$$

$$a_1 = 2 \det \mathbf{R}\{\mathrm{tr}(\mathbf{KR}) - \mathrm{tr}\,\mathbf{K}\,\mathrm{tr}\,\mathbf{R}\} + 2\,\mathrm{tr}\,\mathbf{K}\,\mathrm{tr}\,\mathbf{R}\,\mathrm{tr}(\mathbf{BA}) + \mathrm{tr}(\mathbf{KBRA}) + \mathrm{tr}(\mathbf{RBKA}) - \mathrm{tr}\,\mathbf{R}\{\mathrm{tr}\,\mathbf{BKA})$$
$$+ \mathrm{tr}(\mathbf{KBA})\} - \mathrm{tr}\,\mathbf{K}\{\mathrm{tr}\,\mathbf{BRA}) + \mathrm{tr}(\mathbf{RBA})\}$$

and

$$a_0 = \det \mathbf{A} \det \mathbf{B} + (\det \mathbf{R})^2 - (\mathrm{tr}\,\mathbf{R})^2\,\mathrm{tr}(\mathbf{BA}) + \mathrm{tr}\,\mathbf{R}\{\mathrm{tr}(\mathbf{BRA}) + \mathrm{tr}(\mathbf{RBA})\} - \mathrm{tr}(\mathbf{RBRA}).$$

We conclude that when the matrix of plastic hardening moduli is given by Eq. (39), in the case of multisurface plasticity the loss of ellipticity can be defined as a polynomial equation of order n with respect to H, and the loss of strong ellipticity as a polynomial equation of order $2n$ with respect to H.

4. Critical plastic hardening modulus and critical orientation for single-surface plasticity

4.1. Preliminaries

In this section, the formulae derived previously will be applied to a general isotropic hardening model with smooth plastic potential and yield surface. In particular, for the loss of strong ellipticity an explicit expression of the critical orientation and the critical plastic hardening modulus will be obtained.

In this example, it is assumed that the stress tensor and the unit outward normals to the plastic potential and the yield surface are coaxial. For an isotropic hardening plasticity model, the yield and plastic potential functions are generally expressed in terms of the stress invariants I_σ, J_2 and J_3. (Here I_σ is the first invariant of the stress tensor, $J_2 = 1/2$ (**s**:**s**) and J_3 is the determinant of the deviatoric stress tensor **s**). Then, it follows that:

$$\mathbf{Q} = \frac{1}{\left\|\frac{\partial f}{\partial \boldsymbol{\sigma}}\right\|}\frac{\partial f}{\partial \boldsymbol{\sigma}} = q_1\boldsymbol{\delta} + q_2\mathbf{S} + q_3\mathbf{S}^2$$

and

$$\mathbf{P} = \frac{1}{\left\|\frac{\partial g}{\partial \boldsymbol{\sigma}}\right\|}\frac{\partial g}{\partial \boldsymbol{\sigma}} = p_1\boldsymbol{\delta} + p_2\mathbf{S} + p_3\mathbf{S}^2, \tag{44}$$

where $\mathbf{S} = \mathbf{s}/(\mathbf{s}{:}\mathbf{s})^{1/2}$ is a unit deviatoric stress tensor, and the parameters q_i and p_i (i = 1 to 3) are given in Appendix A. The principal values of the unit deviatoric tensor $\mathbf{S}$ can be written in terms of the single scalar $l \in [0, \pi/3]$ called the Lode angle:

$$S_i = \sqrt{\frac{2}{3}}\cos\left[l - \frac{2}{3}(i-1)\pi\right];\ i \in [1,3]. \tag{45}$$

The ranges of variation of the principal components of $\mathbf{S}$, according to Eq. (45), are as follows: $1/\sqrt{6} \le S_1 \le \sqrt{2/3}$, $-1/\sqrt{6} \le S_2 \le 1/\sqrt{6}$, $-\sqrt{2/3} \le S_3 \le -1/\sqrt{6}$.

Remark 3. In the general noncoaxial case, the tensors $\mathbf{P}$ and $\mathbf{Q}$ can be separated into deviatoric and volumetric parts by introducing the angles φ_P and φ_Q, and the unit deviators $\mathbf{S}_P$ and $\mathbf{S}_Q$ (see Loret, 1992):

$$\mathbf{Q} = \cos\varphi_Q\mathbf{S}_Q + \frac{1}{\sqrt{3}}\sin\varphi_Q\boldsymbol{\delta}$$

and

$$\mathbf{P} = \cos\varphi_P\mathbf{S}_P + \frac{1}{\sqrt{3}}\sin\varphi_P\boldsymbol{\delta}, \tag{46}$$

where $0 \le \varphi_P \le \varphi_Q < \pi/2$. The assumption of deviatoric associativity amounts to postulating that the directions of the deviatoric parts of $\mathbf{P}$ and $\mathbf{Q}$ are identical, namely $\mathbf{S}_P = \mathbf{S}_Q = \hat{\mathbf{S}}$. Thus, the principal axes of the normalized tensors $\mathbf{Q}$ and $\mathbf{P}$ are coaxial, and can be defined in the following forms presented by Loret et al. (1990) and Loret (1992):

$$\mathbf{Q} = \cos\varphi_Q\hat{\mathbf{S}} + \frac{1}{\sqrt{3}}\sin\varphi_Q\boldsymbol{\delta}$$

and

$$\mathbf{P} = \cos\varphi_P\hat{\mathbf{S}} + \frac{1}{\sqrt{3}}\sin\varphi_P\boldsymbol{\delta}. \tag{47}$$

For the cases, $p_3 = q_3 = 0$ or $p_2\, q_3 = p_3\, q_2$ (see Baker and Desai, 1982), the tensors $\mathbf{P}$ and $\mathbf{Q}$ defined by Eq. (44) can be rewritten in a form similar to Eq. (47). However, in general, the tensor $\hat{\mathbf{S}}$ is not equal to $\mathbf{S}$, so any type of anisotropy can be embodied in $\hat{\mathbf{S}}$.

In what follows, linear isotropic elasticity is assumed, so that

$$\mathbf{D}^{\mathrm{e}} = 2G\mathbf{I} + \lambda\boldsymbol{\delta}\otimes\boldsymbol{\delta}, \tag{48}$$

where G and λ are the Lamé constants.

Moreover, we define the deviatoric normal and shear stress in the following form

$$\sigma_n = \mathbf{n} \cdot \mathbf{S} \cdot \mathbf{n}$$

and

$$\tau_n = \sqrt{\mathbf{n} \cdot \mathbf{S}^2 \cdot \mathbf{n} - (\mathbf{n} \cdot \mathbf{S} \cdot \mathbf{n})^2}. \tag{49}$$

Using these quantities, the unit deviatoric stress states can be represented by Mohr's circles [Fig. 1(a)], which are given in a common form

$$\tau_n^2 + \sigma_n^2 + \sigma_n S_i + S_i^2 - \frac{1}{2} = 0, \tag{50}$$

where $i = 1$, 2 and 3 for the first, second and third Mohr's circles, respectively. Now, if we consider a new variable

$$\rho_n = \sigma_n^2 + \tau_n^2, \tag{51}$$

then the Mohr's circle on the plane (σ_n, ρ_n) can be represented by three straight lines [Fig. 1(b)]. Note

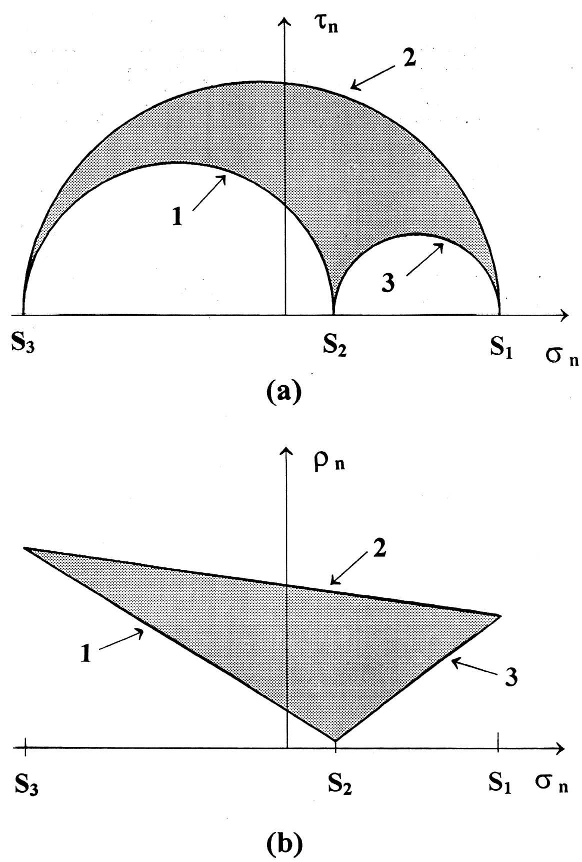


Fig. 1. Schematic representation of the Mohr's circle; (a) on the plane of the unit deviatoric normal and shear stresses, (b) on the plane (σ_n, ρ_n).

that the transformation of the Mohr's circle, using the parameter ρ_n defined above has been introduced by Benallal and Comi (1993), and independently by Szabó (1993).

Furthermore, we introduce the following relationships

$$\mathrm{tr}\,\mathbf{S} = 0,$$

$$\mathrm{tr}\,\mathbf{S}^2 = 1,$$

$$\mathrm{tr}\,\mathbf{S}^3 = 3\,\det\mathbf{S} = \frac{1}{\sqrt{6}}\cos(3l),$$

$$\mathrm{tr}\,\mathbf{S}^4 = \frac{1}{2},$$

$$\mathbf{n}\cdot\mathbf{S}^2\cdot\mathbf{n} = \sigma_n^2 + \tau_n^2 = \rho_n,$$

$$\mathbf{n}\cdot\mathbf{S}^3\cdot\mathbf{n} = \frac{1}{2}\sigma_n + \det\mathbf{S},$$

$$\mathbf{n}\cdot\mathbf{S}^4\cdot\mathbf{n} = \frac{1}{2}\rho_n + \sigma_n\,\det\mathbf{S},$$

$$\mathbf{n}\cdot\mathbf{S}^5\cdot\mathbf{n} = \frac{1}{4}\sigma_n + \left(\frac{1}{2} + \rho_n\right)\det\mathbf{S},$$

$$\det\mathbf{S} = S_i\left(S_i^2 - \frac{1}{2}\right)$$

$$S_iS_j = S_k^2 - \frac{1}{2},$$

and

$$S_i - S_j = \sqrt{2 - 3S_k^2},\ \text{when}\ S_i > S_j. \tag{52}$$

In the derivations of these expressions, the equation $\mathbf{S}^3 - \mathbf{S}/2 - (\det\mathbf{S})\boldsymbol{\delta} = 0$, and the Cayley–Hamilton theorem were employed.

It is convenient to introduce the following general coordinate system with $\mathbf{g}_i$ covariant basis vectors:

$$\mathbf{g}_1 = \mathbf{n},$$

$$\mathbf{g}_2 = \mathbf{S}\cdot\mathbf{n}$$

and

$$\mathbf{g}_3 = \mathbf{S}^2 \cdot \mathbf{n}. \tag{53}$$

The covariant components of the metric tensor are given by

$$(g_{ij}) = \begin{pmatrix} 1 & \sigma_n & \rho_n \\ \sigma_n & \rho_n & \frac{1}{2}\sigma_n + \det \mathbf{S} \\ \rho_n & \frac{1}{2}\sigma_n + \det \mathbf{S} & \frac{1}{2}\rho_n + \sigma_n \det \mathbf{S} \end{pmatrix}. \tag{54}$$

In this coordinate system, the tensor $\mathbf{B}^{\mathrm{e}}$ can be defined with the mixed components as

$$\mathbf{B}^{\mathrm{e}} \equiv \mathbf{B}^{\mathrm{e}}_{\mathrm{sym}} = (\mathbf{B}^{\mathrm{e}})^i_j \mathbf{g}_i \otimes \mathbf{g}^j = GB^i_j \mathbf{g}_i \otimes \mathbf{g}^j, \tag{55}$$

and the vectors **a** and **b** in Eq. (12), using Eqs. (44) and (48), with the contravariant components:

$$\mathbf{a} = 2G\mathbf{n} \cdot \mathbf{P} + \lambda \mathbf{n} \operatorname{tr} \mathbf{P} = 2G\left\{\left[\left(1 + \frac{\lambda}{G}\right)p_1 + \frac{\lambda}{2G}(p_1 + p_3)\right]\mathbf{n} + p_2\mathbf{S} \cdot \mathbf{n} + p_3\mathbf{S}^2 \cdot \mathbf{n}\right\} = 2Ga^i\mathbf{g}_i$$

and

$$\mathbf{b} = 2G\mathbf{Q} \cdot \mathbf{n} + \lambda \mathbf{n} \operatorname{tr} \mathbf{Q} = 2G\left\{\left[\left(1 + \frac{\lambda}{G}\right)q_1 + \frac{\lambda}{2G}(q_1 + q_3)\right]\mathbf{n} + q_2\mathbf{S} \cdot \mathbf{n} + q_3\mathbf{S}^2 \cdot \mathbf{n}\right\} = 2Gb^i\mathbf{g}_i. \tag{56}$$

The coordinates B^i_j in Eq. (55) are defined by

$$B^i_j = \delta^i_j + \frac{1}{1-2\nu}\delta^i_1 g_{1j}, \tag{57}$$

and the contravariant components of vectors a^i and b^k in Eq. (56) can be written as follows:

$$a^1 = \tilde{p}_1 = \frac{1}{1-2\nu}[p_1 + \nu(p_1 + p_3)],$$

$$b^1 = \tilde{q}_1 = \frac{1}{1-2\nu}[q_1 + \nu(q_1 + q_3)],$$

$$a^2 = p_2,$$

$$b^2 = q_2,$$

$$a^3 = p_3$$

and

$$b^3 = q_3. \tag{58}$$

The expression for the plastic hardening modulus, (Eqs. (14) and (21)), using the quantities defined above, can be rewritten as

$$\frac{\mathrm{H}^{\,\mathrm{le}}}{2\mathrm{G}} = 2b^i g_{ik} (B^{-1})^k_j a^{\,j} - c_0 \tag{59}$$

for loss of ellipticity, and

$$\frac{\mathrm{H}^{\,\mathrm{lse}}}{2\mathrm{G}} = b^i g_{ik} (B^{-1})^k_j a^{\,j} + \sqrt{\left[a^i g_{ik} (B^{-1})^k_j a^{\,j}\right]\left[b^l g_{lm} (B^{-1})^m_n b^n\right]} - c_0 \tag{60}$$

for loss of strong ellipticity. In Eqs. (59) and (60), the components $(B^{-1})^i_j$ are defined by

$$(B^{-1})^i_j = \delta^i_j - \frac{1}{2(1-\nu)} \delta^i_1 g_{1j} \tag{61}$$

and

$$\begin{aligned} c_0 &= \mathbf{P}{:}\mathbf{Q} + \frac{\nu}{1-2\nu} \operatorname{tr}\mathbf{P} \operatorname{tr}\mathbf{Q} \\ &= 3p_1q_1 + p_2q_2 + \frac{1}{2}p_3q_3 + p_1q_3 + p_3q_1 + 3 \det\mathbf{S}(p_2q_3 + p_3q_2) + \frac{\nu}{1-2\nu}(3p_1 + p_3)(3q_1 + q_3). \end{aligned} \tag{62}$$

It is noted that the necessary algebraic operations and symbolic manipulations on these tensor quantities defined above can readily be evaluated by using *MATHEMATICA* version 2 (Wolfram, 1991), and *MATHTENSOR* (Parker and Christensen, 1994).

Remark 4. From Eq. (59) and using Eqs. (54), (57) and (58), the plastic hardening modulus can be expressed as

$$\frac{H^{\,\mathrm{le}}}{2G} = A\sigma_n^2 + B\sigma_n\rho_n + C\rho_n^2 + D\sigma_n + E\rho_n + F, \tag{63}$$

where:

$$A = -\frac{1}{1-\nu} p_2 q_2,$$

$$B = -\frac{1}{1-\nu} (p_2 q_3 + p_3 q_2),$$

$$C = -\frac{1}{1-\nu} p_3 q_3,$$

$$D = \frac{1-2\nu}{1-\nu} (\tilde{p}_1 q_2 + p_2 \tilde{q}_1) + (p_2 q_3 + p_3 q_2) + 2 p_3 q_3 \det\mathbf{S},$$

$$E = \frac{1-2\nu}{1-\nu} (\tilde{p}_1 q_3 + p_3 \tilde{q}_1) + 2 p_2 q_2 + p_3 q_3$$

and

$$F = \frac{1-2\nu}{1-\nu} \tilde{p}_1 \tilde{q}_1 + 2(p_2 q_3 + p_3 q_2) \det\mathbf{S} - c_0. \tag{64}$$

The function $H^{\rm le}/2G$ in Eq. (63) can be interpreted as a surface over the plane (σ_n, ρ_n), and corresponds to the localization condition for the general three-dimensional case.

Proposition 1. The function $H^{\rm le}/2G = f(\sigma_n, \rho_n)$ in Eq. (63) is a hyperbolic paraboloid surface in the (σ_n, ρ_n, $H^{\rm le}/2G$) coordinate system.

Proof. Define matrices **T** and **t** as the coefficients of the quadratic form (63):

$$\mathbf{T} = \begin{pmatrix} A & \frac{1}{2}B & 0 & \frac{1}{2}D \\ \frac{1}{2}B & C & 0 & \frac{1}{2}E \\ 0 & 0 & 0 & -\frac{1}{2} \\ \frac{1}{2}D & \frac{1}{2}E & -\frac{1}{2} & F \end{pmatrix}$$

and

$$\mathbf{t} = \begin{pmatrix} A & \frac{1}{2}B & 0 \\ \frac{1}{2}B & C & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For the invariants of the matrix **T**, we obtain

$$\det(\mathbf{T}) = \left[\frac{1}{4(1-\nu)}(p_2q_3 - p_3q_2)\right]^2 > 0$$

and

$$\mathrm{II}_T = -4\det(\mathbf{T}) < 0.$$

The determinant of matrix **t** equals zero and its eigenvalues can be calculated in the following form

$$\lambda_1 = -\frac{1}{2(1-\nu)}\left\{p_2q_2 + p_3q_3 - \sqrt{(p_2q_2 - p_3q_3)^2 + (p_2q_3 + p_3q_2)^2}\right\} > 0,$$

$$\lambda_2 = 0$$

and

$$\lambda_2 = -\frac{1}{2(1-\nu)}\left\{p_2q_2 + p_3q_3 + \sqrt{(p_2q_2 - p_3q_3)^2 + (p_2q_3 + p_3q_2)^2}\right\} < 0.$$

From the invariants of matrices **T** and **t**, and from the eigenvalues of matrix **t**, it immediately follows that the function $H^{\rm le}$ is a hyperbolic paraboloid.

4.2. Critical plastic hardening modulus for the loss of strong ellipticity

We are now concerned with the loss of strong ellipticity, and an explicit expression for the critical hardening modulus and the critical orientation will be derived. This analysis is based on the geometric method which was first proposed by Benallal and Lemaitre (1991) and Benallal (1992), and it was applied in a slightly different form by Benallal and Comi (1993, 1996), Perrin and Leblond (1993), and Szabó (1994, 1997).

The function $H^{\rm lse}/2G$ in Eq. (60) can also be interpreted as a surface over the plane (σ_n, ρ_n), and the quantities σ_n and ρ_n in this function are restricted to the triangular area shown in Fig. 1(b). In Appendix B, it is shown that there is no maximum of the function $H^{\rm lse}$ inside the triangular area. Consequently, the maximum of $H^{\rm lse}$ in question must be above the boundary of the triangular area in Fig. 1(b). Using Eqs. (50) and (51), the sides of triangle are defined by $\rho_n = 1/2 - S_i^2 - S_i\sigma_n$. This expression is substituting into Eqs. (54), (57) and (60), to obtain the following three functions (i = 1,2,3):

$$\frac{H_i^{\rm lse}(\sigma_n,S_i)}{2G} = N_P^i N_Q^i \left\{ (\sigma_n - S_j)(S_k - \sigma_n) + \mu\left(\sigma_n + \frac{M_P^i}{N_P^i}\right)\left(\sigma_n + \frac{M_Q^i}{N_Q^i}\right) + \sqrt{\left[(S_j - \sigma_n)(S_k - \sigma_n) - \mu\left(\sigma_n + \frac{M_P^i}{N_P^i}\right)^2\right]\left[(S_j - \sigma_n)(S_k - \sigma_n) - \mu\left(\sigma_n + \frac{M_Q^i}{N_Q^i}\right)^2\right]} \right\} - c_0, \tag{65}$$

where $\mu=(1-2\nu)/[2(1-\nu)]$,

$$N_P^i = p_2 - S_i p_3,$$

$$N_Q^i = q_2 - S_i q_3,$$

$$M_P^i = \tilde{p}_1 + p_3\left(\frac{1}{2} - S_i^2\right)$$

and

$$M_Q^i = \tilde{q}_1 + q_3\left(\frac{1}{2} - S_i^2\right). \tag{66}$$

The functions $H_1^{\rm lse}/2G$, $H_2^{\rm lse}/2G$ and $H_3^{\rm lse}/2G$ in Eq. (65) are restricted to the sides of the triangle [Fig. 1(b)], respectively. Because the maximum of the function $H_i^{\rm lse}/2G$ in Eq. (65) may be located on the sides or the corner points of the triangle, the critical plastic hardening modulus can be calculated as

$$\frac{H_{\rm crit}^{\rm lse}}{2G} = \max_i \max_{\sigma_n} \frac{H_i^{\rm lse}(\sigma_n,S_i)}{2G};\ i \in [1,3], \tag{67}$$

where $\sigma_n \in [S_3, S_2]$ for i = 1, $\sigma_n \in [S_3, S_1]$ for i = 2 and $\sigma_n \in [S_2, S_1]$ for i = 3.

The necessary condition for a stationary value of the function $H_i^{\rm lse}/2G$ in Eq. (65) is $\partial(H_i^{\rm lse}/2G)/\partial\sigma_n$

$=0$. Using this condition, the stationary value of the normal stress is defined by:

$$\left(\sigma_n^{\rm lse}\right)_{\rm sta}^i = \frac{-U_1 - \sqrt{K_P K_Q}}{U_2}, \tag{68}$$

where

$$U_1 = 2(1-2\nu)\frac{M_P^i}{N_P^i}\frac{M_Q^i}{N_Q^i} - 2(1-\nu)S_i\left(\frac{M_P^i}{N_P^i} + \frac{M_q^i}{N_Q^i}\right) + (3-\nu)S_i^2 - 1,$$

$$U_2 = 2\left(S_i - \frac{M_P^i}{N_P^i} - \frac{M_Q^i}{N_Q^i}\right)$$

and

$$K_I = 2(1-2\nu)\frac{M_I^i}{N_I^i}\left(\frac{M_I^i}{N_I^i} - S_i\right) - (1+\nu)S_i^2 + 1; \tag{69}$$

here, the index I equals P or Q. The stationary value of the normal stress in Eq. (68) may be inside or outside of the associated interval. However, the range of variation of the normal stress σ_n is restricted to the intervals $[S_3,\ S_2]$ for $i = 1$, $[S_3,\ S_1]$ for $i = 2$ and $[S_2,\ S_1]$ for $i = 3$. Thus, its maximum value is defined for the first Mohr circle ($i = 1$) as

$$\left(\sigma_n^{\rm lse}\right)_{\max}^{(1)} = \begin{cases} \left(\sigma_n^{\rm lse}\right)_{\rm sta}^{(1)} & \text{if} \quad S_3 < \left(\sigma_n^{\rm lse}\right)_{\rm sta}^{(1)} \le S_2 \\ S_2 & \text{if} \quad \left(\sigma_n^{\rm lse}\right)_{\rm sta}^{(1)} > S_2 \\ S_3 & \text{if} \quad \left(\sigma_n^{\rm lse}\right)_{\rm sta}^{(1)} < S_3 \end{cases}, \tag{70}$$

for the second Mohr circle ($i = 2$) as

$$\left(\sigma_n^{\rm lse}\right)_{\max}^{(2)} = \begin{cases} \left(\sigma_n^{\rm lse}\right)_{\rm sta}^{(2)} & \text{if} \quad S_3 < \left(\sigma_n^{\rm lse}\right)_{\rm sta}^{(2)} \le S_1 \\ S_1 & \text{if} \quad \left(\sigma_n^{\rm lse}\right)_{\rm sta}^{(2)} > S_1 \\ S_3 & \text{if} \quad \left(\sigma_n^{\rm lse}\right)_{\rm sta}^{(2)} < S_3 \end{cases}, \tag{71}$$

and for the third Mohr circle ($i = 3$) as

$$\left(\sigma_n^{\rm lse}\right)_{\max}^{(3)} = \begin{cases} \left(\sigma_n^{\rm lse}\right)_{\rm sta}^{(3)} & \text{if} \quad S_2 < \left(\sigma_n^{\rm lse}\right)_{\rm sta}^{(3)} \le S_1 \\ S_1 & \text{if} \quad \left(\sigma_n^{\rm lse}\right)_{\rm sta}^{(3)} > S_1 \\ S_2 & \text{if} \quad \left(\sigma_n^{\rm lse}\right)_{\rm sta}^{(3)} < S_2 \end{cases}. \tag{72}$$

Upon the substitution of $(\sigma_n^{\rm lse})_{\max}^i$ into Eq. (65), the corresponding three maximum values of H can be calculated. The critical hardening modulus is defined as the maximum of these functions

$$\frac{H^{\text{lse}}_{\text{crit}}}{2G} = \max_i \frac{H^{\text{lse}}_i\left[\left(\sigma^{\text{lse}}_n\right)^i_{\max}, S_i\right]}{2G}; \ i \in [1,3]. \tag{73}$$

The critical normal stress $(\sigma^{\text{lse}}_n)_{\text{crit}}$ is defined by the one of the three $(\sigma^{\text{lse}}_n)^i_{\max}$, which gives the largest value of the plastic hardening modulus in Eq. (73).

Remark 5. It is important to note that when the tensors **P** and **Q** are symmetric with respect to all principal axes of stress ($P_1 = P_2 = P_3$, and $Q_1 = Q_2 = Q_3$), then, $N^i_P = 0$ and $N^i_Q = 0$. In this case, the critical orientation is indeterminate, and the critical plastic hardening modulus is defined by

$$\frac{H^{\text{lse}}_{\text{crit}}}{2G} = 2\mu M^i_P M^i_Q - c_0 = -\frac{2(1+\nu)}{9(1-\nu)} \operatorname{tr} \mathbf{P} \operatorname{tr} \mathbf{Q} \tag{74}$$

This type of localization mode is called a splitting discontinuity (see Bigoni and Hueckel, 1990a, 1991; Ottosen and Runesson, 1991a). In this case, the loss of ellipticity and loss of strong ellipticity are identical.

Remark 6. In the case of deviatoric associative flow rule Eq. (47), the parameters M^i_P/N^i_P and M^i_Q/N^i_Q in Eqs. (65) and (69), are expressible in the form

$$\frac{M^i_P}{N^i_P} = \frac{1+\nu}{\sqrt{3}(1-2\nu)} \tan \varphi_P,$$

$$\frac{M^i_Q}{N^i_Q} = \frac{1+\nu}{\sqrt{3}(1-2\nu)} \tan \varphi_Q, \tag{75}$$

and $N^i_P = \cos \varphi_P$, $N^i_Q = \cos \varphi_Q$.

In addition, the presented results can easily be applied to the model used by Rudnicki and Rice (1975) by the following relationships:

$$\tan \varphi_P = \sqrt{\frac{2}{3}}\beta$$

and

$$\tan \varphi_Q = \sqrt{\frac{2}{3}}\mu, \tag{76}$$

where the parameters β and μ are defined in the Rudnicki–Rice model (see Rudnicki and Rice, 1975; 3(a)).

Remark 7. For the classical shear band localization (loss of ellipticity), an explicit expression for the critical plastic hardening modulus, using the method considered above, can easily be derived. Because of in the context of shear band analysis, several explicit solutions have been obtained (see, e.g. Bardet, 1990; Benallal, 1992; Benallal and Comi, 1996; Bigoni and Hueckel, 1991; Ottosen and Runesson, 1991a; Runesson et al., 1991), here only as an alternative solution, some simple expressions will be presented. According to Proposition 1, there is no maximum of the function $H^{\text{le}}/2G$ inside the triangular area, therefore, the maximum of this function must be at the boundary of the triangular area in Fig. 1(b). From Eq. (65) (cf. Eqs. (59) and (60)), we obtained the following three functions (i = 1,2,3):

$$\frac{H_i^{\rm le}(\sigma_n,S_i)}{2G} = 2N_P^iN_Q^i\left\{(\sigma_n - S_j)(S_k - \sigma_n) + \frac{(1-2\nu)}{2(1-\nu)}\left(\sigma_n + \frac{M_P^i}{N_P^i}\right)\left(\sigma_n + \frac{M_Q^i}{N_Q^i}\right)\right\} - c_0, \tag{77}$$

where the parameters N_P^i, N_Q^i, M_P^i and M_Q^i are defined by Eq. (66). The necessary condition for the stationary value of the function defined above is $\partial(H_i^{\rm le}/2G)/\partial\sigma_n = 0$. From this condition, the stationary value of the normal stress is defined by

$$\left(\sigma_n^{\rm le}\right)_{\rm sta}^i = \frac{1-2\nu}{2}\left\{\frac{M_P^i}{N_P^i} + \frac{M_Q^i}{N_Q^i}\right\} - (1-\nu)S_i. \tag{78}$$

For the three Mohr circles the normal stress σ_n is restricted to the intervals $[S_3, S_2]$, $[S_3, S_1]$ and $[S_2, S_1]$, respectively. Thus, the maximum values for each of the Mohr circles are defined as being similar to Eqs. (70)–(72). With these quantities, the critical plastic hardening modulus may be expressed according to

$$\frac{H_{\rm crit}^{\rm le}}{2G} = \max_i \frac{H_i^{\rm le}\left[\left(\sigma_n^{\rm le}\right)_{\max}^i, S_i\right]}{2G}; \quad i \in [1,3]. \tag{79}$$

The critical normal stress $(\sigma_n^{\rm le})_{\rm crit}$, then becomes $(\sigma_n^{\rm le})_{\rm crit} = (\sigma_n^{\rm le})_{\max}^i$, where the index i is associated with the largest value of the plastic hardening modulus in Eq. (79).

4.3. Critical orientation for the loss of strong ellipticity

The critical shear band orientation, using a method presented by Benallal and Lemaitre (1991) and Benallal (1992), or another way suggested by Szabó (1994), can also be calculated as

$$\tan^2\theta_{\rm crit}^{\rm lse} = \frac{S_j - \left(\sigma_n^{\rm lse}\right)_{\rm crit}}{\left(\sigma_n^{\rm lse}\right)_{\rm crit} - S_k}, \tag{80}$$

where $j = 2$ and $k = 3$ if $(\sigma_n^{\rm lse})_{\rm crit} = (\sigma_n^{\rm lse})_{\max}^{(1)}$, and $j = 1$ and $k = 3$ if $(\sigma_n^{\rm lse})_{\rm crit} = (\sigma_n^{\rm lse})_{\max}^{(2)}$, and $j = 1$ and $k = 2$ if $(\sigma_n^{\rm lse})_{\rm crit} = (\sigma_n^{\rm lse})_{\max}^{(3)}$.

Note that the critical orientation for the cases of loss of ellipticity (Remark 7) and the Raniecki type comparison solid (2), using $(\sigma_n^{\rm le})_{\rm crit}$ and $(\sigma_n^{R})_{\rm crit}$, can also be obtained from Eq. (80).

Remark 8. Because of the critical orientations calculated from the loss of strong ellipticity condition and according to the Raniecki type comparison solids (see Appendix B) are identical; from Eq. (80), it is evident that the critical normal stresses are also identical:

$$\left(\sigma_n^{\rm lse}\right)_{\rm crit} = \left(\sigma_n^{R}\right)_{\rm crit}.$$

It can readily be checked that Eqs. (68) and (B6), with Eq. (B7), yield the same value of σ_n.

5. Examples

The presented explicit expressions for the critical plastic hardening modulus and the critical orientation can provide different possibilities to compare the loss of ellipticity and loss of strong ellipticity criteria. Here, only two special cases, the case of axially-symmetric extension and axially-symmetric compression, will be investigated.

5.1. Axially-symmetric compression

In this case, the Lode angle equals $\pi/3$ and $S_1 = S_2 = 1/\sqrt{6}$, $S_3 = -\sqrt{2/3}$. Thus, the triangle in Fig. 1(b) collapses to a line, while $\sigma_n \in [S_3, S_1]$. In Eqs. (65)–(69) and Eqs. (77)–(79), the index i equals 2. The parameters M_I^i/N_I^i, using Appendix A, can be expressed by

$$\frac{M_I}{N_I} = \frac{(1+\nu)(\sqrt{2}+\tan\phi_I)}{\sqrt{3}(1-2\nu)(1-\sqrt{2}\tan\phi_I)}, \tag{81}$$

where the index I equals P or Q, and $\phi_I \in [-\pi/2, \pi/2]$; moreover, the angle ψ_I equals $\pi/4$ or $5\pi/4$. The stationary value of normal stress, using Eqs. (68), (78) and (81), can be defined by

$$(\sigma_n)_{\rm sta} = \frac{\sqrt{2}(1+3\nu) + (1-3\nu)(\tan\phi_P + \tan\phi_Q) - 4\sqrt{2}\tan\phi_P\tan\phi_Q}{2\sqrt{3}(1-\sqrt{2}\tan\phi_P)(1-\sqrt{2}\tan\phi_Q)}, \tag{82}$$

for the loss of ellipticity and

$$(\sigma_n)_{\rm sta} = \frac{\frac{1}{\sqrt{3}}\left\{\gamma - \sqrt{2}(3+\nu)(\tan\phi_P + \tan\phi_Q) - 2(1+3\nu)\tan\phi_P\tan\phi_Q\right\} - \sqrt{3a_P a_Q}}{2\left\{2\sqrt{2}\tan\phi_P\tan\phi_Q + 2\nu(\tan\phi_P+\tan\phi_Q) - \sqrt{2}(1+2\nu)\right\}}, \tag{83}$$

for the loss of strong ellipticity condition, where

$$\gamma = 1 - 7\nu - 6\nu^2 \tag{84}$$

and

$$a_I = 1 + \nu + 2\nu^2 + 4\sqrt{2}\nu\tan\phi_I + 2(1-\nu)\tan^2\phi_I. \tag{85}$$

The critical plastic hardening modulus and the critical orientation, using Eqs. (82) and (83) with Eqs. (65), (77), (80) and (81), can be easily calculated. When the critical value of the normal stress is identical to S_1 or S_3 in Eq. (80), the corresponding critical orientation is equal to zero or $\pi/2$, respectively. In these cases, using Eqs. (82) and (83), a relation between ϕ_P and ϕ_Q can be defined. In Table 1, these relationships are summarized for the loss of ellipticity (LE) and the loss of strong ellipticity (LSE).

When the conditions given in Table 1 are upheld, the critical orientation is identical to zero or $\pi/2$. Fig. 2 shows these domains on the (ϕ_P, ϕ_Q) plane for $\nu = 0.3$. In Fig. 2(a) for the loss of ellipticity, the

Table 1
Axially-symmetric compression

	$(\sigma_n)_{\rm crit} = S_3 = -\sqrt{2/3}$ $\tan\theta_{\rm crit} = \sqrt{\pi/2}$ $\tan\phi_Q \le$	$(\sigma_n)_{\rm crit} = S_1 = 1/\sqrt{6}$ $\tan\theta_{\rm crit} = 0^\circ$ $\tan\phi_Q \ge$
LE	$\sqrt{2} - \tan\phi_P$	$-\frac{\sqrt{2}\nu + (1-\nu)\tan\phi_P}{1-\nu-2\sqrt{2}\tan\phi_P}$
LSE	$-\frac{2\sqrt{2}\nu}{1-\nu} - \tan\phi_P$	$\frac{\sqrt{2}\nu + (1+\nu^2)\tan\phi_P}{1+\nu^2+2\sqrt{2}\nu\tan\phi_P}$

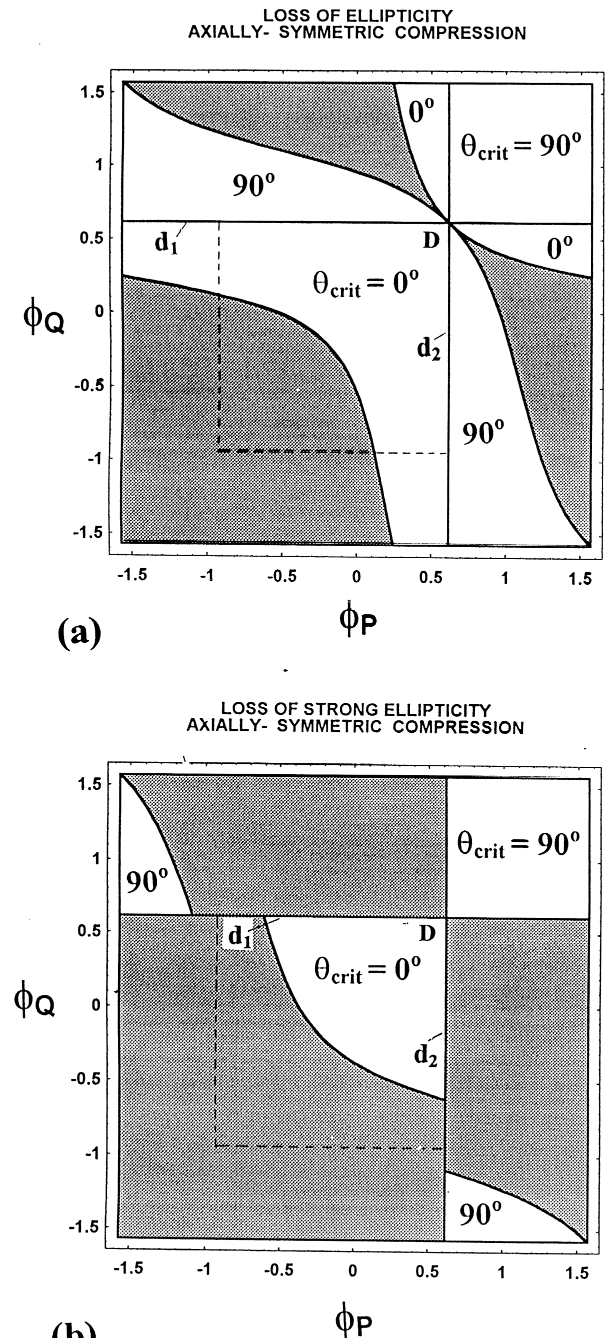


Fig. 2. Comparison of the limit values of the critical orientation for axially-symmetric compression on the plane (ϕ_P, ϕ_Q), (a) loss of ellipticity, (b) loss of strong ellipticity.

variation of the critical orientation on the shaded area, using Eqs. (80) and (82), can be calculated by

$$\tan^2\theta_{\rm crit} = \frac{2 + (1+3\nu)(\tan\phi_P + \tan\phi_Q) - 2(2+3\nu)\tan\phi_P\tan\phi_Q}{(5-3\nu)(\tan\phi_P + \tan\phi_Q) - 2(1-3\nu)\tan\phi_P\tan\phi_Q - 8}. \tag{86}$$

In the case of loss of strong ellipticity, the critical orientation from Eqs. (80) and (83) can be obtained.

The lines d_1 and d_2 on Fig. 2 represent discontinuities which correspond to the singularity of Eq. (81) $(1-\sqrt{2}\tan\phi_I = 0)$, while the point D corresponds to the splitting discontinuity.

Remark 9. Note that when the deviatoric asociative flow rule Eq. (47) is applied to the relation between ϕ_I and φ_I, using Eqs. (75) and (81), the relation can be defined by

$$\tan\phi_I = \frac{\tan\varphi_I - \sqrt{2}}{1 + \sqrt{2}\tan\varphi_I}. \tag{87}$$

For the condition $0 \le \varphi_P \le \varphi_Q < \pi/2$, the deviatoric associativity can be seen enclosed by the dotted line in Fig. 2, which was recently discussed by Szabó (1997).

5.2. Axially-symmetric tension

In the case of axially-symmetric tension, the Lode angle equals zero, $S_3 = S_2 = -1/\sqrt{6}$ and $S_1 = \sqrt{2/3}$. The normal stress, similar to the compression case, defined on the interval $\sigma_n \in [S_3, S_1]$, and the parameters M_I^i/N_I^i, using Appendix A, are defined by

$$\frac{M_I}{N_I} = \frac{(1+\nu)(1+2\tan\psi_I)}{\sqrt{6}(1-2\nu)(1-\tan\psi_I)}, \tag{88}$$

where the index I equals P or Q, and $\psi_I \in [0, 2\pi]$. The stationary value of the normal stress, using Eqs. (68), (78) and (88), can be defined by

$$(\sigma_n)_{\rm stat} = \frac{4 - (1-3\nu)(\tan\psi_P + \tan\psi_Q) - 2(1+3\nu)\tan\psi_P\tan\psi_Q}{2\sqrt{6}(1-\tan\psi_P)(1-\tan\psi_Q)} \tag{89}$$

for the loss of ellipticity and

$$(\sigma_n)_{\rm stat} = \frac{-\sqrt{\frac{2}{3}}\{(1+3\nu) + (3+\nu)(\tan\psi_P + \tan\psi_Q) - \gamma\tan\psi_P\tan\psi_Q\} - \sqrt{6g_Pg_Q}}{4\{(1+2\nu)\tan\psi_P\tan\psi_Q - \nu(\tan\psi_P + \tan\psi_Q) - 1\}}, \tag{90}$$

for the loss of strong ellipticity condition, where

$$g_I = 1 - \nu + 4\nu\tan\psi_I + (1+\nu+2\nu^2)\tan^2\psi_I. \tag{91}$$

The conditions when the critical orientation equals zero or $\pi/2$ are given in Table 2, and are illustrated in Fig. 3.

Fig. 3. shows the critical domains in the (ϕ_P, ϕ_Q) plane for $\nu = 0.3$. Note that this figure was made from the relationships defined in Table 2 using the $\psi_I = \sin^{-1}\tan\phi_I$ transformation.

In the case of loss of ellipticity, when $S_3 < (\sigma_n)_{crit} < S_1$, the critical orientation can be calculated by

$$\tan^2\theta_{\text{crit}} = \frac{(1+\nu)(\tan\psi_P + \tan\psi_Q - 2\tan\psi_P\tan\psi_Q)}{(1-\nu)(\tan\psi_P + \tan\psi_Q) + 2\nu\tan\psi_P\tan\psi_Q - 2} \tag{92}$$

while the critical orientation for the loss of strong ellipticity from Eqs. (80) and (90) can be obtained.

Remark 10. In the case of deviatoric asociative flow rule Eq. (47), the relation between ψ_I and φ_I, using Eqs. (75) and (88), may be defined by

$$\tan\psi_I = \frac{\sqrt{2}\tan\varphi_I - 1}{2 + \sqrt{2}\tan\varphi_I}. \tag{93}$$

According to this relation, when $\varphi_I = 0$ then $\phi_I = -24.09°$, while $\varphi_I \to \pi/2$ then $\phi_I \to 35.2644°$, which corresponds to the d_1 or d_2 lines in Fig. 3.

6. Conclusion

The main objective of this paper was the analysis of the loss of strong ellipticity condition within the framework of multisurface and single-surface elastoplasticity in small deformation range. The results obtained can be summarized as follows.

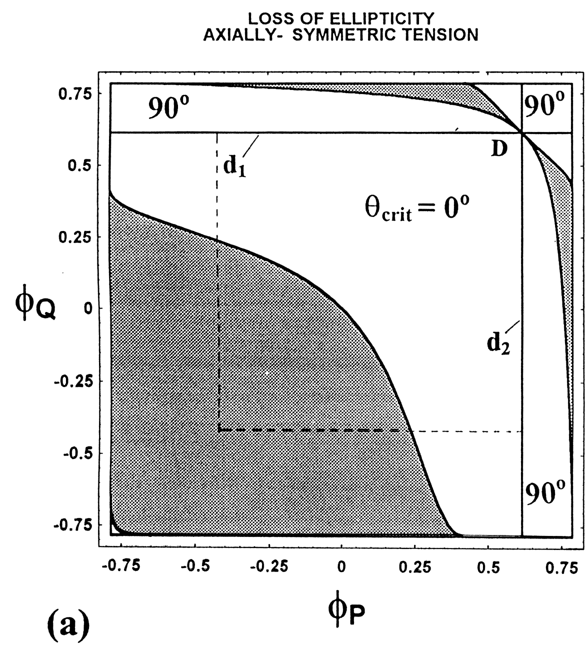
First, a generalization of the Sherman–Morrison formula for the case where a multiple rank one updates was presented. Using this formula, some explicit expressions for the determinant of the acoustic tensor and its symmetric part were derived for the multisurface plasticity models. These expressions are associated to the loss of ellipticity and loss of strong ellipticity. Moreover, an explicit criterion for uniqueness was derived. In this context the determinant of the symmetric part of the fourth-order constitutive tangent tensor was analysed. As an illustrative example, explicit expressions for the plastic hardening modulus in the case of double-surface plasticity with a one parameter family of hardening moduli have been presented.

Next, the loss of strong ellipticity has been investigated for single-surface plasticity. An explicit expression for the critical plastic hardening modulus and the critical orientation, using the same geometric method, has been derived for both Hill's and Raniecki's comparison solids. These expressions are valid for a general nonassociative flow rule, and it is assumed that the principal axes of the normalized tensors **P** and **Q** (unit outward normals to the plastic potential and yield surfaces, respectively) are coaxial.

As noted already, Bigoni and Zaccaria (1992a) have given an analytical solution for the plastic hardening modulus in the case of Raniecki's comparison solids, which is also a solution for the Hill type comparison solids (i.e. in their work, it is shown that the loss of strong ellipticity occurs simultaneously in the Hill's comparison solids and the best chosen Raniecki's comparison solid). However, it should be emphasized that the results presented in this paper based on a completely different method.

Table 2
Axially-symmetric tension

	$(\sigma_n)_{\text{crit}} = S_3 = -1/\sqrt{6}$ $\tan\theta_{\text{crit}} = \pi/2$ $\tan\psi_Q \leq$	$(\sigma_n)_{\text{crit}} = S_1 = \sqrt{\frac{2}{3}}$ $\tan\theta_{\text{crit}} = 0°$ $\tan\psi_Q \geq$
LE	$\frac{2-\tan\psi_P(1-\nu)}{2\nu\tan\psi_P+1-\nu}$	$\frac{\tan\psi_P}{2\tan\psi_P-1}$
LSE	$-\frac{2\nu+(1+\nu^2)\tan\psi_P}{1+\nu^2+2\nu\tan\psi_P}$	$\frac{(\nu-1)\tan\psi_P}{1-\nu+4\nu\tan\psi_P}$



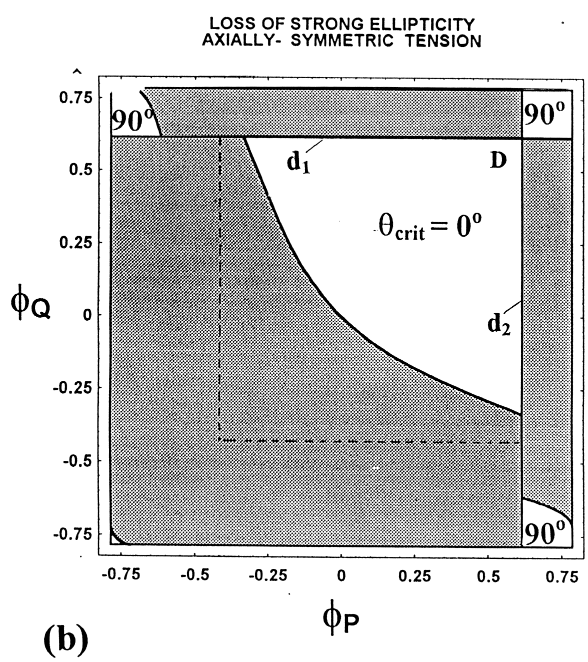


Fig. 3. Comparison of the limit values of the critical orientation for axially-symmetric tension on the plane (ϕ_P, ϕ_Q), (a) loss of ellipticity, (b) loss of strong ellipticity.

In addition, an analysis of the onset of shear band localization (loss of ellipticity) has been presented. Although these results (presented in Remark 7) are well known in the literature, the method used in this paper provides a proper possibility to compare them with the loss of strong ellipticity.

In particular, as an illustrative example, the critical orientation for the loss of strong ellipticity and the classical shear band localization (loss of ellipticity) were compared for axially-symmetric compression and tension.

Finally, it is concluded that the presented expressions for the loss of strong ellipticity can be used to predict the onset of shear banding, due to the bounding nature of the loss of strong ellipticity over the loss of ellipticity region.

Acknowledgements

This research has been supported by the National Development and Research Foundation, Hungary (under Contract: OTKA, T 023929). This support is gratefully acknowledged. The author is also grateful to Dr. Steven M. Christensen (MathTensor, Inc., USA, http://smc.vnet.net/mathtensor.html) for helping to use of *MATHTENSOR*.

Appendix A

The parameters in Eq. (44) are given by

$$p_1 = \frac{1}{p_0}\left(\frac{\partial g}{\partial I_\sigma} - \frac{2}{3}J_2\frac{\partial g}{\partial J_3}\right),$$

$$q_1 = \frac{1}{q_0}\left(\frac{\partial f}{\partial I_\sigma} - \frac{2}{3}J_2\frac{\partial f}{\partial J_3}\right),$$

$$p_2 = \frac{1}{p_0}\sqrt{2J_2}\frac{\partial g}{\partial J_2},$$

$$q_2 = \frac{1}{q_0}\sqrt{2J_2}\frac{\partial f}{\partial J_2},$$

$$p_3 = \frac{1}{p_0}2J_2\frac{\partial g}{\partial J_3}$$

and

$$q_3 = \frac{1}{q_0}2J_2\frac{\partial f}{\partial J_3},$$

where

$$p_0 = \sqrt{3\left(\frac{\partial g}{\partial I_0}\right)^2 + 2J_2\left(\frac{\partial g}{\partial J_2}\right)^2 + 6J_3\frac{\partial g}{\partial J_2}\frac{\partial g}{\partial J_3} + \frac{2}{3}(J_2)^2\left(\frac{\partial g}{\partial J_3}\right)^2}$$

and

$$q_0 = \sqrt{3\left(\frac{\partial f}{\partial I_0}\right)^2 + 2J_2\left(\frac{\partial f}{\partial J_2}\right)^2 + 6J_3\frac{\partial f}{\partial J_2}\frac{\partial f}{\partial J_3} + \frac{2}{3}(J_2)^2\left(\frac{\partial f}{\partial J_3}\right)^2},$$

Because, **P** and, **Q** are unit tensors, namely $\mathbf{P}:\mathbf{P} = 1$ and $\mathbf{Q}:\mathbf{Q} = 1$, the principal components of these tensors can be expressed in the following form

$$P_1 = \cos\phi_P \cos\psi_P,$$

$$Q_1 = \cos\phi_Q \cos\psi_Q,$$

$$P_2 = \cos\phi_P \sin\psi_P,$$

$$Q_2 = \cos\phi_Q \sin\psi_Q,$$

$$P_3 = \sin\phi_P$$

and

$$Q_3 = \sin\phi_Q, \tag{A1}$$

where ϕ_P and $\phi_Q \in [-\pi/2, \pi/2]$, and ψ_P and $\psi_Q \in [0, 2\pi]$.

From Eq. (44), using the definitions above, it follows that

$$\begin{pmatrix} \cos\phi_P \cos\psi_P \\ \cos\phi_P \sin\psi_P \\ \sin\phi_P \end{pmatrix} = \begin{pmatrix} 1 & S_1 & S_1^2 \\ 1 & S_2 & S_2^2 \\ 1 & S_3 & S_3^2 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}.$$

The matrix defined above is known as the Vandermonde matrix. In the general case, when the Lode angle $l \in (0, \pi/3)$ ($l \neq 0$ and $l \neq \pi/3$), the parameters p_i can be expressed as

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \frac{1}{d}\begin{pmatrix} S_2S_3(S_2 - S_3) & S_1S_3(S_3 - S_1) & S_1S_2(S_1 - S_2) \\ S_3^2 - S_2^2 & S_1^2 - S_3^2 & S_2^2 - S_1^2 \\ S_2 - S_3 & S_3 - S_1 & S_1 - S_2 \end{pmatrix} \begin{pmatrix} \cos\phi_p \cos\psi_P \\ \cos\phi_P \sin\psi_P \\ \sin\phi_P \end{pmatrix}, \tag{A2}$$

where $d = (S_1 - S_2)(S_2 - S_3)(S_1 - S_3)$. The parameters q_i can be calculated in a similar way. When $l = 0$ or $l = \pi/3$, the expressions presented above are simplified. From $\mathbf{P}:\mathbf{P} = 1$, using Eq. (44), we obtain

$$3p_1^2 + 2p_1p_3 + p_2^2 + 6p_2p_3 \det \mathbf{S} + \frac{1}{2}p_3^2 - 1 = 0.$$

When the Lode angle $l \in (0, \pi/3)$ ($l \neq 0$ and $l \neq \pi/3$), this equation is represented by an ellipsoid surface in the (p_1, p_2, p_3) coordinate system, and when $l = 0$ ($S_2 = S_3 = -1/\sqrt{6}$, $S_1 = \sqrt{2/3}$) or $l = \pi/3$ ($S_1 = S_2 = 1/\sqrt{6}$, $S_3 = -\sqrt{2/3}$), it is represented by an elliptic cylindrical surface. In this last case, the parameter p_1 may be choosen arbitralily, and

$$\left.\begin{aligned} p_2 &= \frac{1}{\sqrt{6}}\cos\phi_P\cos\psi_P - 2\sqrt{\frac{2}{3}}\sin\phi_P + \sqrt{\frac{3}{2}}p_1 \\ p_3 &= \cos\phi_P\cos\psi_P + 2\sin\phi_P - 3p_1 \end{aligned}\right\} \text{ for } l = 0,\ (\phi_P = \tan^{-1}(\sin\psi_P))$$

and

$$\left.\begin{aligned} p_2 &= 2\sqrt{\frac{2}{3}}\cos\phi_P\cos\psi_P - \frac{1}{\sqrt{6}}\sin\phi_P - \sqrt{\frac{3}{2}}p_1 \\ p_3 &= 2\cos\phi_P\cos\psi_P + \sin\phi_P - 3p_1 \end{aligned}\right\} \text{ for } l = \frac{\pi}{3},\ \left(\psi_P = \frac{\pi}{4} \text{ or } \frac{5\pi}{4}\right).$$

Finally, we note that when $P_1 = P_2 = P_3$ and $Q_1 = Q_2 = Q_3$, the angles $\psi_P = 45^\circ$ and $\phi_P = \tan^{-1}(1/\sqrt{2})$, and $\psi_Q = 45^\circ$ and $\phi_Q = \tan^{-1}(1/\sqrt{2})$.

Appendix B

In this Appendix, an explicit expression for the critical plastic hardening modulus for the Raniecki-type comparison solids will be derived. The constitutive tangent tensor for the family of comparison solids introduced by Raniecki (1979) is defined as

$$\mathbf{D}_R^{ep} = \mathbf{D}^e - \frac{\mathbf{D}^e{:}\mathbf{R} \otimes \mathbf{R}{:}\mathbf{D}^e}{4\psi(H + \mathbf{Q}{:}\mathbf{D}^e{:}\mathbf{P})}, \tag{B1}$$

where ψ is a free parameter, and the tensor $\mathbf{R}$ is given by

$$\mathbf{R} = \mathbf{P} + \psi\mathbf{Q}.$$

For this model, using Eq. (14) or Eq. (21), the plastic hardening modulus can be expressed as

$$H^R = \frac{1}{4\psi}[\mathbf{R}(\psi){:}\mathbf{D}^e \cdot \mathbf{n}] \cdot \mathbf{B}^{e-1} \cdot [\mathbf{n} \cdot \mathbf{D}^e{:}\mathbf{R}(\psi)] - \mathbf{Q}{:}\mathbf{D}^e{:}\mathbf{P}$$

and the critical plastic hardening modulus corresponds to the solution of the constrained maximization problem:

$$H^R_{crit} = \max_{\mathbf{n}} \min_{\psi} H^R(\mathbf{n},\psi),$$

subject to $|\mathbf{n}| = 1$, and $\psi \in R^+$.

Bigoni and Zaccaria (1992a) proved that the extremal problem defined above can be rewritten as

$$H^R_{crit} = \min_{\psi} \max_{\mathbf{n}} \mathrm{H}^R(\mathbf{n},\psi), \tag{B2}$$

subject to $|\mathbf{n}| = 1$, and $\psi \in R^+$. Moreover, they have shown that the critical plastic hardening modulus is identical for the Raniecki and the Hill type comparison solids, namely

$$H^{lse}_{crit} = H^R_{crit}, \tag{B3}$$

or

$$\max_{\mathbf{n}} H^{\mathrm{lse}}(\mathbf{n}) = \min_{\psi} \max_{\mathbf{n}} H^{R}(\mathbf{n},\psi),$$

where $H^{\mathrm{lse}}_{\mathrm{crit}}$ is the critical plastic hardening modulus for the loss of the strong ellipticity condition in the Hill type comparison solid (see Eq. (4), and $H^{\mathrm{lse}}(\mathbf{n})$ is given by Eq. (21).

Now, the maximization problem in Eq. (B2) can be solved, using the method presented in Remark 4, by making the following identifications $p_i \rightarrow (p_i/\psi + q_i)/4$ and $q_i \rightarrow (p_i + \psi q_i)$. It can easily be shown that for an associative constitutive model, the function $H^R(\sigma_n, \rho_n, \psi)/2G$ is a parabolic cylinder surface in the $(\sigma_n, \rho_n, H^R/2G)$ coordinate system. It is obvious that the parameter ψ does not effect the type of this surface. Hence, the maximum of function $H^R(\sigma_n, \rho_n, \psi)$ may be located on the boundary of the triangle [see Fig. 1(b)].

The boundary lines of the triangle, using Eqs. (50) and (51), can be defined as

$$\rho_n = 1/2 - S_i^2 - S_i\sigma_n, \tag{B4}$$

where index i corresponds to the first (i = 1), the second (i = 2), and the third (i = 3) Mohr circle, respectively.

From Eq. (63), using Eq. (B4) and the indentifications defined above, we obtained the following three functions (i = 1,2,3):

$$\frac{H_i^R(\sigma_n,\psi,S_i)}{2G} = \frac{\left(N^i_P + \psi N^i_Q\right)^2}{2\psi}\left[(\sigma_n - S_j)(S_k - \sigma_n) + \frac{1-2\nu}{2(1-\nu)}\left(\sigma_n + \frac{M^i_P + \psi M^i_Q}{N^i_P + \psi N^i_Q}\right)^2\right] - c_0, \tag{B5}$$

where the parameters N^i_P, N^i_Q, M^i_P and M^i_Q are defined in Eq. (66).

The critical plastic hardening modulus is defined by the solution of the problem

$$\frac{H^R_{\mathrm{crit}}}{2G} = \max_i \min_{\psi} \max_{\sigma_n} \frac{H_i^R(\sigma_n,\psi,S_i)}{2G};\ i \in [1,3].$$

For the solution of the maximization problem with respect to σ_n, we obtain:

$$\frac{\partial\left[H_i^R(\sigma_n,\psi,S_i)/2G\right]}{\partial\sigma_n} = 0 \Longrightarrow \left(\sigma_n^R\right)^i_{\mathrm{sta}} = (1-2\nu)\frac{M^i_P + \psi M^i_Q}{N^i_P + \psi N^i_Q} - (1-\nu)S_i. \tag{B6}$$

Let us introduce the notation

$$\frac{\tilde{H}_i^R(\psi,S_i)}{2G} \equiv \frac{H_i^R\left[\left(\sigma_n^R\right)^i_{\mathrm{sta}},\psi,S_i\right]}{2G}.$$

Then, the minimization problem for $\tilde{H}_i^R(\psi,S_i)/2G$ with respect to ψ can be solved. The stationary value of ψ is given by

$$\frac{\partial\left[\tilde{H}_i^R(\psi,S_i)/2G\right]}{\partial\psi} = 0 \Longrightarrow \psi^i_{\mathrm{sta}} = \sqrt{\frac{2(1-2\nu)M^i_P\left(M^i_P - N^i_P S_i\right) - N^{i2}_P\left[S_i^2(1+\nu) - 1\right]}{2(1-2\nu)M^i_Q\left(M^i_Q - N^i_Q S_i\right) - N^{i2}_Q\left[S_i^2(1+\nu) - 1\right]}}. \tag{B7}$$

Finally, the critical plastic hardening modulus is defined by the maximum of the three functions, $\tilde{H}_i^R(\psi^i_{\mathrm{sta}},S_i)/2G$, namely:

$$\frac{H^R_{\text{crit}}}{2G} = \max_i \frac{\tilde{H}^R_i(\psi^i_{\text{sta}}, S_i)}{2G};\ i \in [1,3].$$

It can now be noted that the critical plastic hardening modulus, $H^R_{\text{crit}}/2G$, is located on the sides or the corner points of the triangle. From Eq. (B3), it is easy to conclude that the critical plastic hardening modulus for the family of comparison solid analysed in this paper, is also located on the boundary of the triangle. In other words, the maximum of $H^{\text{lse}}/2G$ in Eq. (60) must be above the sides (or the corner points) of the triangular area in Fig. 1(b).

Note that the loss of strong ellipticity condition for the Raniecki type comparison solids was analyzed by Bigoni and Zaccaria (1992a). They presented an analytical solution for the critical plastic hardening modulus, however, the result presented herein is more explicit, and it was derived in a different way. Nonetheless, the two different solutions provide identical numerical values.

Finally, it is interesting to investigate the Raniecki type comparison solid in the context of loss of uniqueness. The general bifurcation criterion is first satisfied when the determinant of the symmetric part of the tangent modulus tensor is equal to zero (see, e.g. Ottosen and Runesson, 1991a; Neilsen and Schreyer, 1993; Ottosen and Ristinmaa, 1996). From this condition, an expression for the plastic hardening modulus can be obtained. In the case of Hill type comparison solid, using the symmetric part of $\mathbf{D}^{\text{ep}}$ defined by Eq. (4), the plastic hardening modulus is given by

$$\det \mathbf{D}^{\text{ep}}_{\text{sym}} = 0 \rightarrow H^g_{(\text{Hill})} = \frac{1}{2}\left\{\sqrt{(\mathbf{Q}{:}\mathbf{D}^{\text{e}}{:}\mathbf{Q})(\mathbf{P}{:}\mathbf{D}^{\text{e}}{:}\mathbf{P})} - \mathbf{Q}{:}\mathbf{D}^{\text{e}}{:}\mathbf{P}\right\}. \tag{B8}$$

This expression has previously been obtained, e.g., by Raniecki and Bruhns (1981), Runesson and Mroz (1989), Bigoni and Hueckel (1991) and Neilsen and Schreyer (1993).

In the case of Raniecki's comparison solid, using the tensor $\mathbf{D}^{\text{ep}}_R$ defined by Eq. (B1), the plastic hardening modulus can be expressed as a function of ψ:

$$\det \mathbf{D}^{\text{ep}}_R = 0 \Longrightarrow H^g_{(\text{Raniecki})}(\psi) = \frac{1}{4\psi}\mathbf{P}{:}\mathbf{D}^{\text{e}}{:}\mathbf{P} + \frac{1}{4}\psi\mathbf{Q}{:}\mathbf{D}^{\text{e}}{:}\mathbf{Q} - \frac{1}{2}\mathbf{Q}{:}\mathbf{D}^{\text{e}}{:}\mathbf{P}. \tag{B9}$$

Then, the critical plastic hardening modulus is defined by

$$\left(H^g_{(\text{Raniecki})}\right)_{\text{crit}} = \inf_{\psi} H^g_{(\text{Raniecki})}(\psi).$$

The solution for the stationary value of ψ is obtained as

$$\frac{\partial\left[H^g_{(\text{Raniecki})}(\psi)\right]}{\partial\psi} = 0 \Longrightarrow \psi_{\text{crit}} = \sqrt{\frac{\mathbf{P}{:}\mathbf{D}^{\text{e}}{:}\mathbf{P}}{\mathbf{Q}{:}\mathbf{D}^{\text{e}}{:}\mathbf{Q}}}. \tag{B10}$$

Substituting ψ_{crit} into Eq. (B9), the critical plastic hardening modulus takes the same form as Eq. (B8):

$$\left(H^g_{(\text{Raniecki})}\right)_{\text{crit}} = H^g_{(\text{Raniecki})}(\psi_{\text{crit}}) = \frac{1}{2}\left\{\sqrt{(\mathbf{Q}{:}\mathbf{D}^{\text{e}}{:}\mathbf{Q})(\mathbf{P}{:}\mathbf{D}^{\text{e}}{:}\mathbf{P})} - \mathbf{Q}{:}\mathbf{D}^{\text{e}}{:}\mathbf{P}\right\}. \tag{B11}$$

This result was first established by Raniecki (1979).

References

Asaro, R.J., 1983. Crystal plasticity. J. Appl. Mech. 50, 921–934.

Baker, R., Desai, C.S., 1982. Consequences of deviatoric normality in plasticity with isotropic strain hardening. Int. J. Numer. Anal. Methods Geomech. 6, 383–390.

Bardet, J.P., 1990. A comprehensive review of strain localization in elastoplastic soils. Comput. and Geotechnics 10, 163–188.

Benallal, A., Lemaitre, J. 1991. Localization phenomena in thermo–elasto–plasticity. In: Zyczkowski, M. (Ed.), Creep in Structures, IUTAM Symp. Cracow Poland 1990. Springer–Verlag, Berlin Heidelberg, pp. 223–235.

Benallal, A., 1992. On localization phenomena in thermo–elasto–plasticity. Arch. Mech. 44, 15–29.

Benallal, A., Comi, C., 1993. Explicit Solutions to Localisation Conditions via a Geometrical Method, Part 1 — The Coaxial Case. Laboratoire de Mechanique at Technologie, (E.N.S. Cachan/C.N.R.S./Universite Paris 6), France Report No. 143, February.

Benallal, A., Comi, C., 1996. Localization analysis via a geometrical method. Int. J. Solids Struct. 33, 99–119.

Bigoni, D., Hueckel, T., 1990a. A note on strain localization for a class of non-associative plasticity rules. Ingenieur-Archive 60, 491–499.

Bigoni, D., Hueckel, T., 1990b. On uniqueness and strain localization in plane strain and plane stress elastoplasticity. Mech. Res. Comm. 17, 15–23.

Bigoni, D., Hueckel, T., 1991. Uniqueness and localization — I. Associative and nonassociative elasto–plasticity. Int. J. Solids Struct. 28, 197–213.

Bigoni, D., Zaccaria, D., 1992a. Loss of strong ellipticity in non-associative elastoplasticity. J. Mech. Phys. Solids 40, 1313–1331.

Bigoni, D., Zaccaria, D., 1992b. Strong ellipticity of comparison solids in elastoplasticity with volumetric non-associativity. Int. J. Solids Struct. 29, 2123–2136.

Bigoni, D., Zaccaria, D., 1993. On strain localization analysis of elastoplastic materials at finite strain. Int. J. Plasticity 9, 21–33.

Borré, G., Maier, G., 1989. On linear versus nonlinear flow rules in strain localization analysis. Meccanica 24, 36–41.

Dao, M., Asaro, R.J., 1996. Localized deformation modes and non-Schmid effects in crystalline solids. Part I. Critical conditions of localization. Mech. of Materials 23, 71–102.

Doghri, I., Billardon, R., 1995. Investigation of localization due to damage in elasto–plastic materials. Mech. of Materials 23, 71–102.

Duszek, M.K., Perzyna, P., 1991. The localization of plastic deformation in thermoplastic solids. Int. J. Solids Struct. 27, 1419–1443.

Hadamard, J., 1903. Leçons Sur la Propagation des Ondes et les Equations de l'Hydrodynamique. Hermann, Paris.

Hill, R., 1962. Acceleration waves in solids. J. Mech. Phys. Solids. 10, 1–16.

Hutchinson, J.W., 1970. Elastic–plastic behaviour of polycrystalline metals and composites. Proc. R. Soc. London A318, 247–272.

Ichikawa, Y., Ito, T., Mroz, Z., 1990. A strain localization condition applying multi-response theory. Ingenieur-Archiv 60, 542–552.

Koiter, W.T., 1960. General theorems of elasto–plastic solids. In: Sneddon, I.N., Hill, R. (Eds.), Progress in Solid Mechanics, vol. 1. North Holland Publishing Company, pp. 167–221.

Loret, B., Prévost, J.H., Harireche, O., 1990. Loss of hyperbolicity in elastic–plastic solids with deviatoric associativity. Eur. J. Mech., A/Solids 9, 225–231.

Loret, B., 1992. Does deviation from deviatoric associativity lead to the onset of flutter instability? J. Mech. Phys. Solids. 40, 1363–1375.

Mandel, J., 1962. Ondes plastiques dans un milieu indéfini à trois dimensions. Journal de Mécanique 1, 3–30.

Mandel, J., 1965. Géneralisation de la théorie de plasticité de W.T. Koiter. Int. J. Solids Struct. 1, 273–295.

Molenkamp, F., 1985. Comparison of frictional material models with respect to shear band initiation. Géotechnique 35, 127–143.

Needleman, A., Tvergaard, V., 1992. Analyses of plastic flow localization in metals. Appl. Mech. Rev. 45, 3–18.

Neilsen, M.K., Schreyer, H.L., 1993. Bifurcations in elastic–plastic materials. Int. J. Solids Struct. 30, 521–544.

Ottosen, N.S., Ristinmaa, M., 1996. Corners in plasticity — Koiter's theory revisited. Int. J. Solids Struct. 33, 3697–3721.

Ottosen, N.S., Runesson, K., 1991a. Properties of discontinuous bifurcation solutions in elasto–plasticity. Int. J. Solids Struct. 27, 401–421.

Ottosen, N.S., Runesson, K., 1991b. Acceleration waves in elasto–plasticity. Int. J. Solids Struct. 28, 135–159.

Parker, L., Christensen, S.M., 1994. MATHTENSOR®, A System for Doing Tensor Analysis by Computer. Addison–Wesley Publ. Comp, New York.

Peirce, D., 1983. Shear band bifurcations in ductile single crystals. J. Mech. Phys. Solids. 31, 133–153.

Perrin, G., Leblond, J.B., 1993. Rudnicki and Rice's analysis of strain localization revisited. J. of Applied Mechanics 60, 842–846.

Petryk, H., 1997. Plastic instability: criteria and computational approaches. Archives of Computational Methods in Engineering 4, 111–151.

Prévost, J.H., 1984. Localization of deformations in elastic–plastic solids. Int. J. Numer. Anal. Methods Geomech. 8, 187–196.

Raniecki, B., 1979. Uniqueness criteria in solids with non-associated plastic flow laws at finite deformations. Bull. Acad. Pol. Sci. Sér. Sci. Techniques. 27, 391–399.
Raniecki, B., Bruhns, O.T., 1981. Bounds to bifurcation stresses in solids with non-associated plastic flow law at finite strain. J. Mech. Phys. Solids. 29, 153–172.
Reddy, B.D., Gültop, T., 1995. Acceleration waves in finitely deformed elastic–plastic solids. Eur. J. Mech., A/Solids 14, 529–551.
Rice, J.R. 1976. The localization of plastic deformation. In: Koiter, W.T. (Ed.), Theoretical and Applied Mechanics, Proc. 14th IUTAM Congress, Delft, The Netherlands, 30 August–4 September 1976. North–Holland, Amsterdam, pp. 207–220.
Rice, J.R., Rudnicki, J.W., 1980. A note on some features of the theory of localization of deformation. Int. J. Solids Struct. 16, 597–605.
Rizzi, E., Maier, G., Willam, K., 1996. On failure indicators in multidissipative materials. Int. J. Solids Struct. 33, 3187–3214.
Rudnicki, J.W., Rice, J.R., 1975. Conditions for the localization of deformation in pressure-sensitive dilatant materials. J. Mech. Phys. Solids 23, 371–394.
Runesson, K., Mroz, Z., 1989. A note on nonassociated plastic flow rules. Int. J. Plasticity 5, 639–658.
Runesson, K., Ottosen, N.S., Peric, D., 1991. Discontinuous bifurcations of elastic–plastic solutions at plane stress and plane strain. Int. J. Plasticity 7, 99–121.
Sawischlewski, E., Steinmann, P., Stein, E., 1996. Modelling and computation of instability phenomena in multisurface elasto–plasticity. Comput. Mech. 18, 245–258.
Sewell, M.J., 1973. A yield-surface corner lowers the buckling stress of an elastic–plastic plate under compression. J. Mech. Phys. Solids 21, 19–45.
Sewell, M.J., 1974. A plastic flow rule at a yield vertex. J. Mech. Phys. Solids 22, 469–490.
Simo, J.C., Kennedy, J.G., Govindjee, S., 1988. Non-smooth multisurface plasticity and viscoplasticity. Loading/unloading conditions and numerical algorithms. Int. J. Num. Meth. Eng. 26, 2161–2185.
Steinmann, P., 1996. On localization analysis in multisurface hyperelasto–plasticity. J. Mech. Phys. Solids 44, 1691–1713.
Steinmann, P., Larsson, R., Runesson, K., 1997. On the localization properties of multiplicative hyperelasto–plastic continua with strong discontinuities. Int. J. Solids Struct. 34, 969–990.
Szabó, L., 1985. Evaluation of elastic–viscoplastic tangent matrices without numerical inversion. Computers and Structures 21, 1235–1236.
Szabó, L. 1993. Shear band localization of elasto–plastic solids at finite strain. In: Plasticity '93, The Fourth Intl. Symp. on Plasticity and its Current Application 19–23 July, 1993, Baltimore, U.S.A.
Szabó, L., 1994. Shear band formulations in finite strain elastoplasticity. Int. J. Solids Struct. 31, 1291–1308.
Szabó, L., 1997. On the eigenvalues of the fourth-order constitutive tensor and loss of strong ellipticity in elastoplasticity. Int. J. Plasticity 13, 809–835.
Wolfram, S., 1991. Mathematica, A System for Doing Mathematics by Computer, 2nd ed. Addison–Wesley, Redwood City, California.
Yatomi, C., Yashima, A., Iizuka, A., Sano, I., 1989. General theory of shear bands formation by a non-coaxial cam–clay model. Soils and Foundations 29, 41–53.
Zbib, H.M., Aifantis, E.C., 1988. On the localization and post-localization behavior plastic deformation. I. On the initiation of shear bands. Res. Mech. 23, 261–277.
Zbib, H.M., 1989. Deformations of materials exhibiting noncoaxiality and finite rotations. Scripta Metallurgica 23, 789–794.
Zbib, H.M., 1991. On the mechanics of large inelastic deformations: noncoaxiality, axial effects in torsion and localization. Acta Mech. 87, 179–196.
Zbib, H.M., 1993. On the mechanics of large inelastic deformations: kinematics and constitutive modeling. Acta Mech. 96, 119–138.